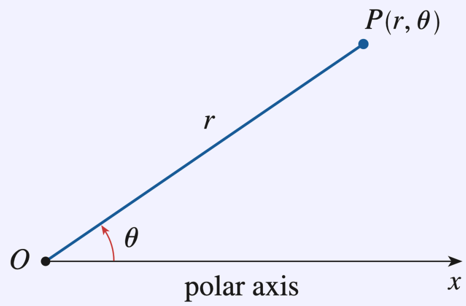
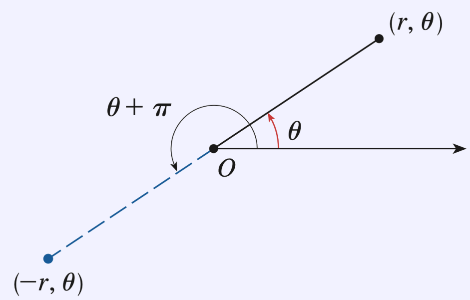
10.3 Polar Coordinates

Definition. The polar coordinate system is defined by:

- A point called the **pole** (or origin), denoted by O.
- A ray starting from O, called the **polar axis**, usually drawn horizontally to the right.

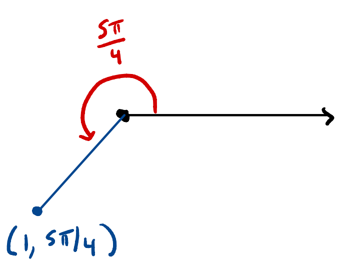


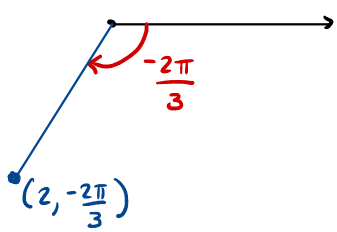


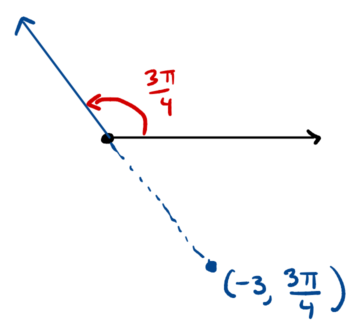
- A point P in the plane is represented by the ordered pair (r, θ), where:
  - r is the distance from O to P.
  - θ is the angle (in radians) between the polar axis and the line OP.
- An angle is positive if measured counterclockwise and negative if measured clockwise.
- If $r < 0$, the point (r, θ) lies on the opposite side of the pole, at the same distance $|r|$.

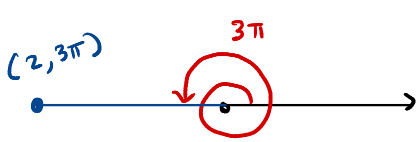
Example. Plot the points with the following polar coordinates:

(a) $(1, \frac{5\pi}{4})$ (b) $(2, 3\pi)$ (c) $(2, -\frac{2\pi}{3})$ (d) $(-3, \frac{3\pi}{4})$

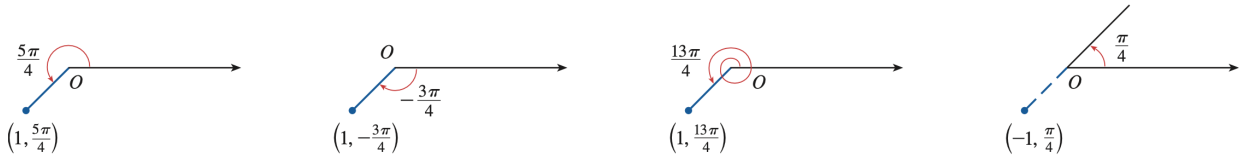




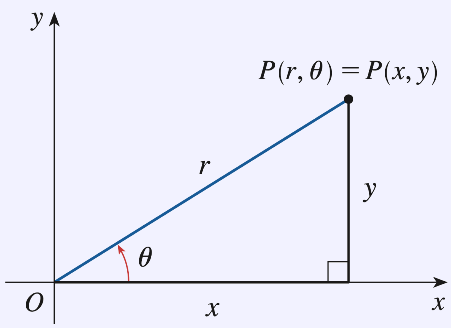




Remark. In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$.



Theorem. The relationship between polar and Cartesian coordinates is given by:



$$x = r\cos\theta$$

$$y = r\sin\theta$$

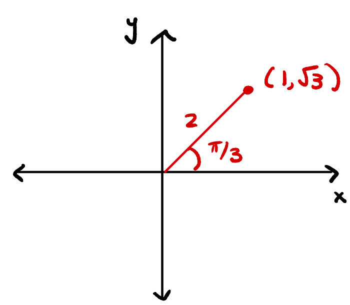
$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

Example. Convert the point $(2, \frac{\pi}{3})$ from polar to Cartesian coordinates.

$$x = r\cos\theta = 2\cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

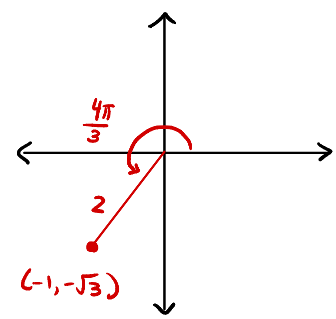


The Cartesian coordinates are $(1, \sqrt{3})$

Example. Convert the Cartesian point $(-1, -\sqrt{3})$ to polar coordinates.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) = \tan^{-1}(\sqrt{3}) \leftarrow \frac{\pi}{3}$$



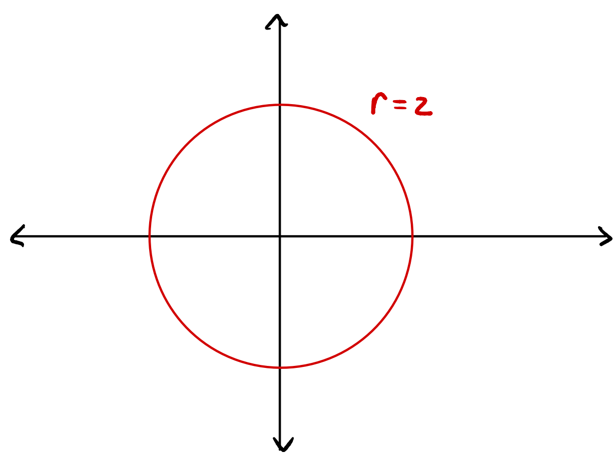
Adjust for the correct quadrant: $\theta = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$

The polar coordinates are $\left(2, \frac{4\pi}{3}\right)$

Definition. The graph of a polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Example. What curve is represented by the polar equation $r = 2$?

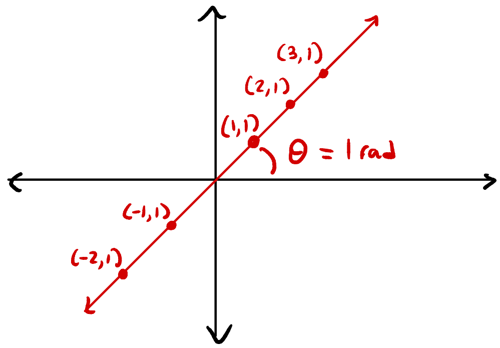
The curve consists of all points (r, θ) with $r = 2$



In general, the equation $r = a$ represents a circle with radius $|a|$.

Example. Sketch the polar curve $\theta = 1$.

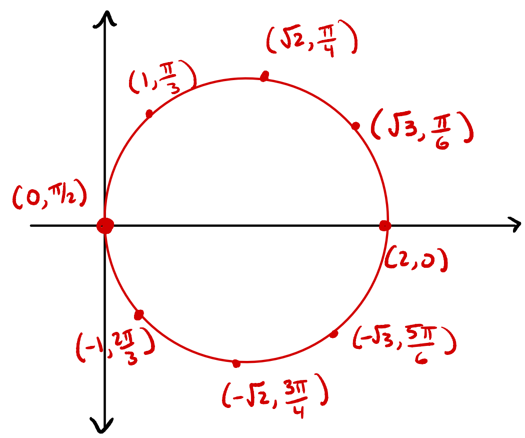
The curve consists of all points (r, θ) where the polar angle is 1 radian



Note: The points (r,θ) on the line with $r>0$ are in the first quadrant, whereas those with $r<0$ are in the third quadrant.

Example. Sketch the curve $r = 2\cos\theta$ and convert it to a Cartesian equation.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2



From $x = r\cos\theta$, we have $\frac{x}{r} = \cos\theta$

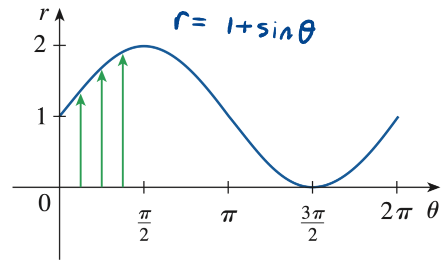
$r = 2\cos\theta \Rightarrow r = 2\cdot\frac{x}{r} \Rightarrow r^2 = 2x \Rightarrow x^2+y^2 = 2x$

Completing the square, $x^2 - 2x + y^2 = 0 \Rightarrow x^2 - 2x + 1 + y^2 = 1$

$\Rightarrow (x-1)^2 + y^2 = 1$, a circle centered at $(1,0)$ with radius 1.

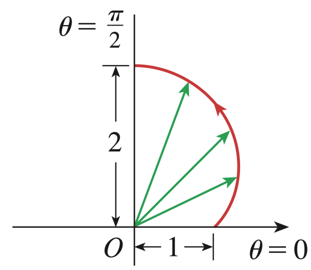
Example. Sketch the curve $r = 1 + \sin\theta$.

- In Cartesian coordinates, graph the behavior of r for $0 \le \theta \le 2\pi$.



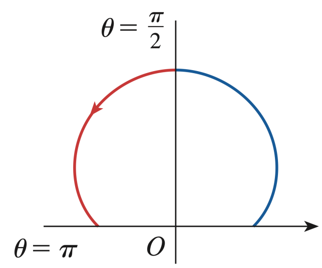
- Analyze the behavior of r for $0 \le \theta \le \pi/2$.

r increases from 1 to 2



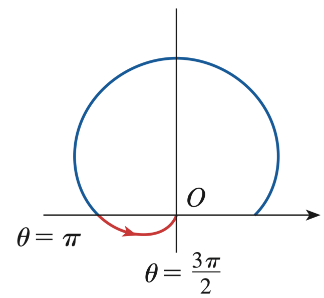
- Analyze the behavior of r for $\pi/2 \le \theta \le \pi$.

r decreases from 2 to 1



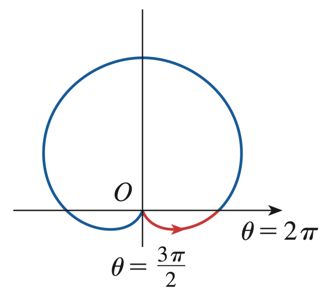
- Analyze the behavior of r for $\pi \le \theta \le 3\pi/2$.

r decreases from 1 to 0



- Analyze the behavior of r for $3\pi/2 \le \theta \le 2\pi$.

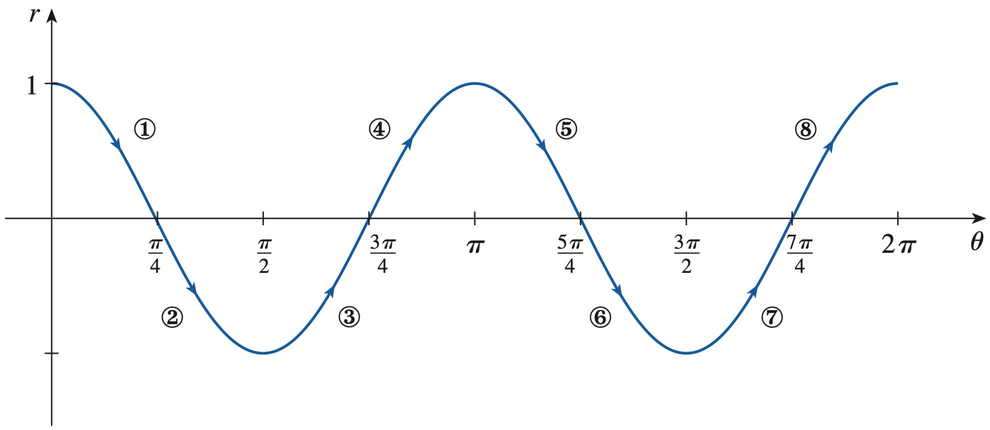
r increases from 0 to 1



This is called a cardioid

Example. Sketch the curve $r = \cos 2\theta$.

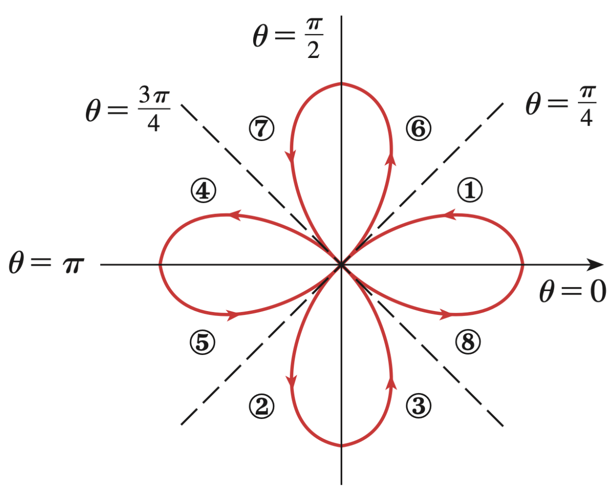
- Sketch $r = \cos 2\theta$ for $\theta \in [0, 2\pi]$. This helps us visualize the values of r as θ varies.



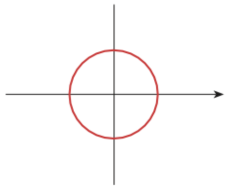
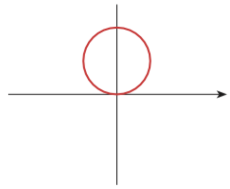
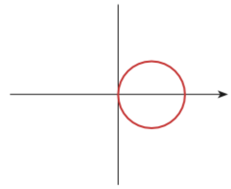
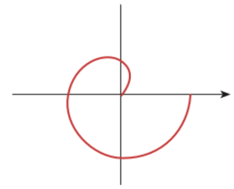
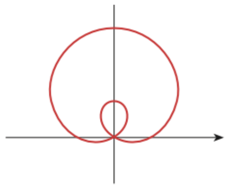
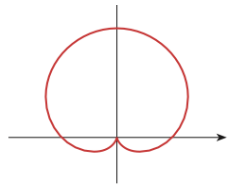
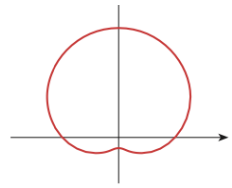
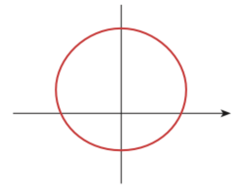
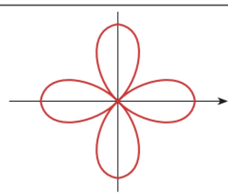
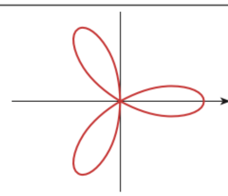
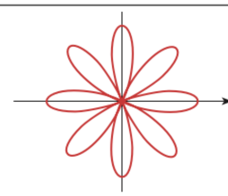
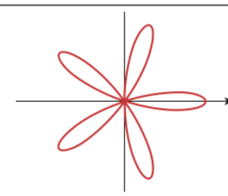
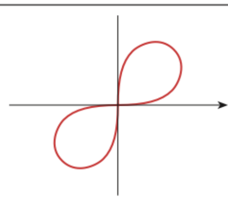
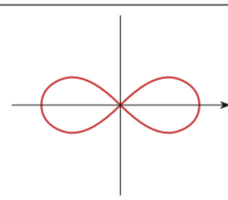
- Analyze the value of r for varying values of θ.

$0 \le \theta \le \frac{\pi}{4}$	$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$	$\frac{\pi}{2} \le \theta \le \frac{3\pi}{4}$	$\frac{3\pi}{4} \le \theta \le \pi$
r decreases from 1 to 0 Quadrant 1	r decreases from 0 to −1 Quadrant 3	r increases from −1 to 0 Quadrant 4	r increases from 0 to 1 Quadrant 2

- The resulting curve will have four loops, forming a four-leaved rose.



Common Polar Curves

Circles and Spiral	 $r = a$ circle	 $r = a \sin \theta$ circle	 $r = a \cos \theta$ circle	 $r = a\theta$ spiral
Limaçons $r = a \pm b \sin \theta$ $r = a \pm b \cos \theta$ $(a > 0, b > 0)$ Orientation depends on the trigonometric function (sine or cosine) and the sign of b	 $a < b$ limaçon with inner loop	 $a = b$ cardioid	 $a > b$ dimpled limaçon	 $a \geqslant 2b$ convex limaçon
Roses $r = a \sin n\theta$ $r = a \cos n\theta$ n-leaved if n is odd $2n$-leaved if n is even	 $r = a \cos 2\theta$ four-leaved rose	 $r = a \cos 3\theta$ three-leaved rose	 $r = a \cos 4\theta$ eight-leaved rose	 $r = a \cos 5\theta$ five-leaved rose
Lemniscates Figure-eight-shaped curves	 $r^2 = a^2 \sin 2\theta$ lemniscate	 $r^2 = a^2 \cos 2\theta$ lemniscate		