10.2 Tangents to Parametric Curves (Solutions)

1. Consider the parametric equations given by

$$x = t^2 + 1, \quad y = 2t - 3.$$

The slope of the curve is determined by the ratio of the derivatives with respect to t:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Differentiating the component functions yields

$$\frac{dx}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = 2.$$

Substituting these into the slope formula, we obtain

$$\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}.$$

To find the slope at the specific parameter value $t = 1$, we substitute directly:

$$\frac{dy}{dx} = \frac{1}{1} = 1.$$

2. For the curve defined by

$$x = \sin t, \quad y = \cos t,$$

horizontal tangents occur where the derivative $\frac{dy}{dt}$ is zero, provided that $\frac{dx}{dt}$ is non-zero. First, we find the values of t for which $\frac{dy}{dt} = 0$:

$$\frac{dy}{dt} = -\sin t = 0 \implies t = 0, \pi, 2\pi.$$

Next, we verify that $\frac{dx}{dt} \neq 0$ at these values. Calculating $\frac{dx}{dt} = \cos t$:

$$\text{At } t = 0, 2\pi, \quad \cos t = 1 \neq 0,$$
$$\text{At } t = \pi, \quad \cos t = -1 \neq 0.$$

The points of horizontal tangency correspond to these t values:

$$(0, 1) \quad \text{and} \quad (0, -1).$$

3. Let the curve be given by

$$x = e^t, \quad y = e^{-t}.$$

The slope of the tangent line is derived as follows:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-e^{-t}}{e^t} = -e^{-2t}.$$

At $t = 0$, the slope m is

$$m = -e^{-2(0)} = -1.$$

To find the equation of the tangent line, we determine the point of tangency at $t = 0$:

$$x = e^0 = 1, \quad y = e^{-0} = 1.$$

Using the point-slope form $y - y_1 = m(x - x_1)$ with the point $(1, 1)$ and slope -1:

$$y - 1 = -1(x - 1) \implies y = -x + 2.$$

4. Consider the curve

$$x = t - \sin t, \quad y = 1 - \cos t.$$

The slope is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1 - \cos t}.$$

Substituting $t = \frac{\pi}{4}$ into the expression yields

$$\frac{dy}{dx} = \frac{\sin(\pi/4)}{1 - \cos(\pi/4)} = \frac{\frac{\sqrt{2}}{2}}{1 - \frac{\sqrt{2}}{2}}.$$

Multiplying the numerator and denominator by 2 simplifies the fraction to

$$\frac{\sqrt{2}}{2 - \sqrt{2}}.$$

Rationalizing the denominator gives the final slope:

$$\frac{\sqrt{2}(2 + \sqrt{2})}{4 - 2} = \frac{2\sqrt{2} + 2}{2} = \sqrt{2} + 1.$$

5. For the parametric equations

$$x = \ln t, \quad y = t^2,$$

the derivative is calculated as

$$\frac{dy}{dx} = \frac{2t}{1/t} = 2t^2.$$

When $t = 1$, the slope is $m = 2(1)^2 = 2$. The corresponding point on the curve is

$$x = \ln 1 = 0, \quad y = 1^2 = 1.$$

The equation of the tangent line passing through $(0, 1)$ with slope 2 is

$$y - 1 = 2(x - 0) \implies y = 2x + 1.$$

6. Given the curve

$$x = 3t^2 + 2, \quad y = 4t^3 - 5,$$

we compute the general slope formula:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12t^2}{6t} = 2t \quad (\text{assuming } t \neq 0).$$

Evaluating this at $t = -1$ gives

$$\frac{dy}{dx} = 2(-1) = -2.$$

7. Consider the curve defined by

$$x = t^2 + 2t, \quad y = 3t - 1.$$

The slope is given by

$$\frac{dy}{dx} = \frac{3}{2t + 2}.$$

At $t = 2$, the slope becomes

$$m = \frac{3}{2(2) + 2} = \frac{3}{6} = \frac{1}{2}.$$

The coordinate of the point of tangency at $t = 2$ is

$$x = 2^2 + 2(2) = 8, \quad y = 3(2) - 1 = 5.$$

Finally, the equation of the tangent line at $(8, 5)$ is

$$y - 5 = \frac{1}{2}(x - 8) \implies y = \frac{1}{2}x + 1.$$