10.2 Calculus with Parametric Curves

Theorem. Let $x = f(t)$ and $y = g(t)$ be a parametric curve, where f and g are differentiable functions of t. If y is a differentiable function of x, the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided that } \frac{dx}{dt} \neq 0.$$

Proof.

By the chain rule, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

If $\frac{dx}{dt} \neq 0$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

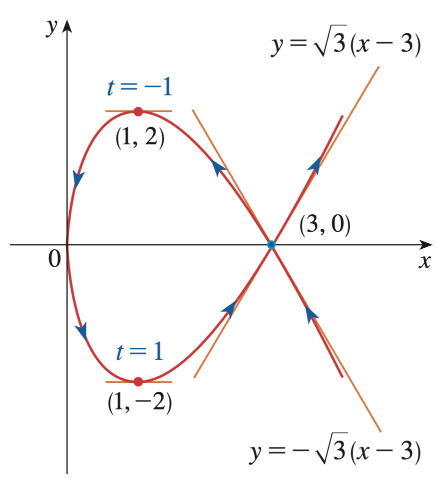
$\square$

Example. A curve C is defined by the parametric equations $x = t^2, y = t^3 - 3t$.

(a) Show that C has two tangents at the point $(3, 0)$ and find their equations.

(b) Find the points on C where the tangent is horizontal or vertical.

(a) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$



$\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 3t^2 - 3$

Hence

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

If $x = 3$, then $3 = t^2 \Rightarrow t = \pm\sqrt{3}$ → (And both of these correspond to the point $(3, 0)$)

If $t = \sqrt{3}$:

$$\frac{dy}{dx} = \frac{3(\sqrt{3})^2 - 3}{2\sqrt{3}} = \frac{6}{2\sqrt{3}} = \sqrt{3}$$

The tangent line is $\boxed{y - 0 = \sqrt{3}(x-3)}$

We are at the point $(3,0)$

If $t = -\sqrt{3}$

$$\frac{dy}{dx} = \frac{3(-\sqrt{3})^2 - 3}{2(-\sqrt{3})} = \frac{6}{-2\sqrt{3}} = -\sqrt{3}$$

The tangent line is $\boxed{y - 0 = -\sqrt{3}(x-3)}$

(b) Horizontal tangents occur when $\frac{dy}{dx} = 0 \Rightarrow dy/dt = 0$ and $dx/dt \neq 0$.

$$\frac{dy}{dt} = 3t^2 - 3 = 0 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$$

If $t = 1$, $x = 1^2 = 1$

$y = 1^3 - 3(1) = -2$

Point: $(1, -2)$

Plug t back into the parametric equations to get the point (x,y)

If $t = -1$, $x = (-1)^2 = 1$

$y = (-1)^3 - 3(-1) = 2$

Point: $(1, 2)$

Vertical tangents occur when $dx/dt = 0$

$2t = 0 \Rightarrow t = 0$

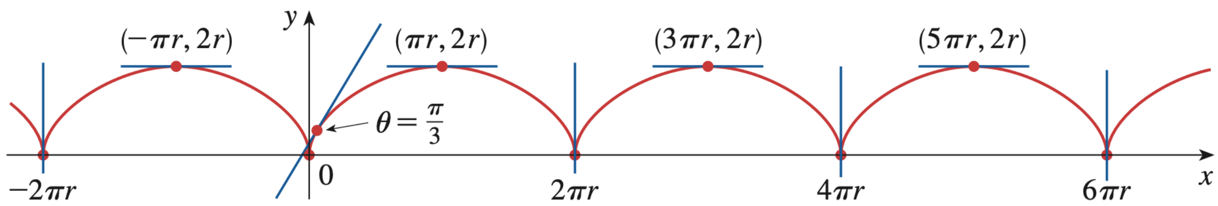
If $t = 0$, $x = 0^2 = 0$ and $y = 0^3 - 3(0) = 0$

Point: $(0,0)$

Example.

(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$.

(b) At what points is the tangent horizontal? When is it vertical?



(a) $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{r\sin\theta}{r - r\cos\theta} = \dfrac{\sin\theta}{1-\cos\theta}$ for $1 - \cos\theta \neq 0$

When $\theta = \pi/3$,

$$\frac{dy}{dx} = \frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1-1/2} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

The point corresponding to $\theta = \pi/3$ is

$$x = r(\pi/3 - \sin\pi/3) = r(\pi/3 - \sqrt{3}/2)$$

$$y = r(1 - \cos\pi/3) = r(1 - 1/2) = \frac{r}{2}$$

$$\boxed{y - \frac{r}{2} = \sqrt{3}\left(x - r(\pi/3 - \sqrt{3}/2)\right)}$$

(b) Horizontal tangents occur when $\frac{dy}{dx} = 0$, which happens when $\underset{dy/d\theta}{\sin\theta = 0}$ and $\underset{dx/d\theta}{1 - \cos\theta \neq 0}$

$\sin\theta = 0 \Rightarrow \theta = 0, \pi, 2\pi, \ldots \Rightarrow \theta = n\pi$ for integers n

$\cos\theta \neq 1$ rules out even multiples of π

$\theta = (2n-1)\pi$ for integers n [odd multiples of π]

The corresponding points are

$$x = r\left((2n-1)\pi - \cancelto{0}{\sin\left((2n-1)\pi\right)}\right) = (2n-1)\pi \cdot r$$

$$y = r\left(1 - \underbrace{\cos\left((2n-1)\pi\right)}_{-1}\right) = 2r$$

$$\boxed{\left((2n-1)\pi r, 2r\right)}$$