# 10.1 Parametric Curves (Solutions)

1. Sketch and identify the curve.

$$x = t^2 - 4t, \qquad y = t + 2, \qquad 0 \le t \le 5.$$

**Solution.** Eliminate the parameter. From $y = t + 2$, we have $t = y - 2$. Substitute into $x$:

$$x = (y-2)^2 - 4(y-2) = (y-4)^2 - 4.$$

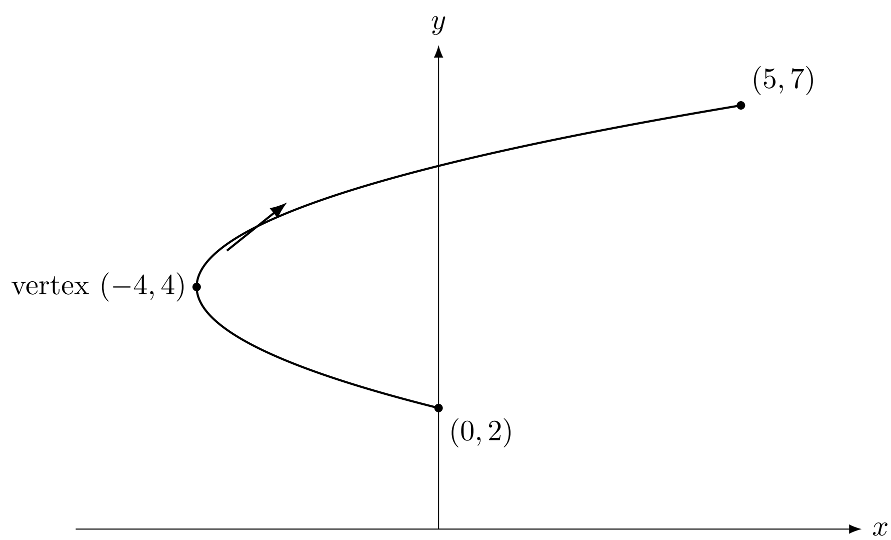
So the curve is the parabola

$$x = (y-4)^2 - 4,$$

which opens to the *right* with vertex at $(-4, 4)$. Because $0 \le t \le 5$, we have $2 \le y = t + 2 \le 7$, so we only trace the portion with $2 \le y \le 7$. Key points:

$$t = 0: \ (x, y) = (0, 2), \qquad t = 2: \ (x, y) = (-4, 4), \qquad t = 5: \ (x, y) = (5, 7).$$

As $t$ increases, $y$ increases, so the motion is upward along the parabola (from $(0, 2)$ toward the vertex and then out to $(5, 7)$).



2. Sketch and identify the curve.

$$x = 2\cos t - 1, \qquad y = 2\sin t + 3, \qquad 0 \le t \le 2\pi.$$

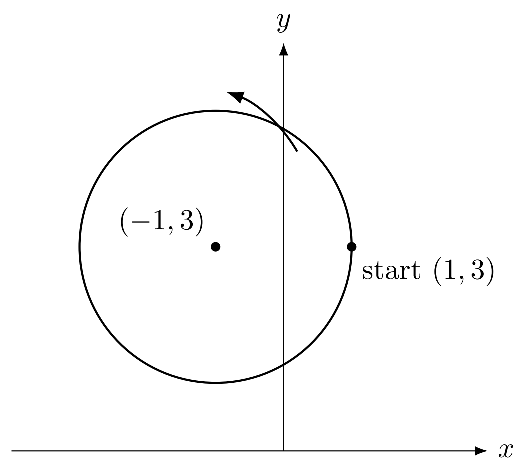
**Solution.** Rewrite as

$$x + 1 = 2\cos t, \qquad y - 3 = 2\sin t.$$

Square and add:

$$(x+1)^2 + (y-3)^2 = 4(\cos^2 t + \sin^2 t) = 4.$$

So the curve is a circle centered at $(-1, 3)$ with radius 2. At $t = 0$, the point is $(x, y) = (1, 3)$ (the rightmost point), and as $t$ increases the motion is counterclockwise.



3. Sketch and identify the curve.

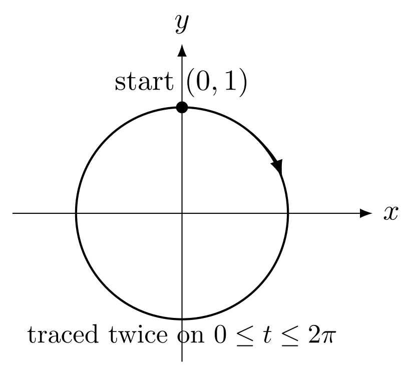
$$x = \sin(2t), \qquad y = \cos(2t), \qquad 0 \le t \le 2\pi.$$

**Solution.** Eliminate the parameter:

$$x^2 + y^2 = \sin^2(2t) + \cos^2(2t) = 1,$$

so the curve is the unit circle centered at the origin.

Let $\theta = 2t$. As $t$ runs from 0 to $2\pi$, $\theta$ runs from 0 to $4\pi$, so the circle is traced *twice*. At $t = 0$, $(x, y) = (0, 1)$. For small $t > 0$, $x = \sin(2t) > 0$ and $y = \cos(2t) < 1$, so the motion initially goes to the right from the top point, meaning the direction is *clockwise*. Thus the unit circle is traced twice clockwise.



4. Sketch and identify the curve.

$$x = t^3, \qquad y = t, \qquad -2 \le t \le 2.$$

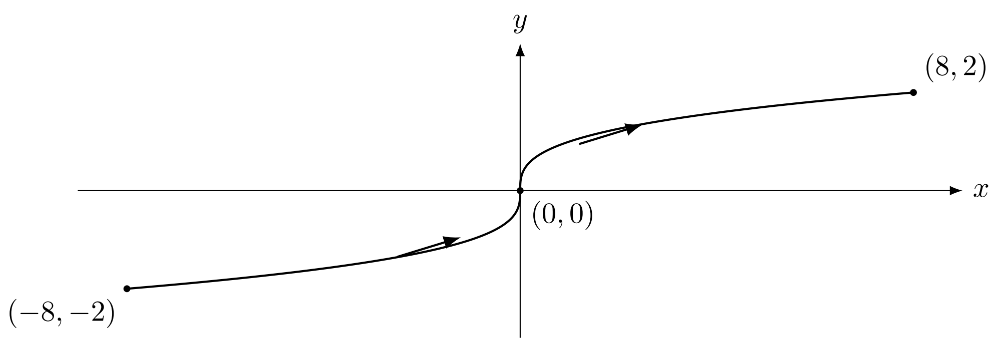
**Solution.** Since $y = t$, substitute $t = y$ into $x = t^3$:

$$x = y^3.$$

So the curve is the cubic $x = y^3$ (equivalently $y = \sqrt[3]{x}$). The parameter interval $-2 \le t \le 2$ gives

$$t = -2: \ (x, y) = (-8, -2), \qquad t = 0: \ (0, 0), \qquad t = 2: \ (8, 2).$$

As $t$ increases, the motion goes from $(-8, -2)$ through $(0, 0)$ to $(8, 2)$.



5. Sketch and identify the curve.

$$x = \sin t, \qquad y = \sin^2 t, \qquad 0 \le t \le 2\pi.$$

**Solution.** Eliminate the parameter:

$$y = \sin^2 t = (\sin t)^2 = x^2.$$

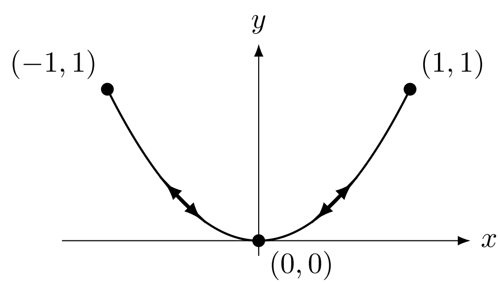
So the curve lies on the parabola $y = x^2$. Since $x = \sin t$, we have $-1 \le x \le 1$. Therefore the curve is

$$y = x^2, \qquad -1 \le x \le 1.$$

Direction/motion (key times):

$$t = 0 : (0,0), \quad t = \frac{\pi}{2} : (1,1), \quad t = \pi : (0,0), \quad t = \frac{3\pi}{2} : (-1,1), \quad t = 2\pi : (0,0).$$

So the particle goes from $(0,0)$ up the *right* branch to $(1,1)$, returns to $(0,0)$, then goes up the *left* branch to $(-1,1)$, and returns again to $(0,0)$.



6. Find a parametrization $\mathbf{r}(t) = (x(t), y(t))$ of the line segment from

$$P_0 = (-2, 1) \quad \text{to} \quad P_1 = (4, -3),$$

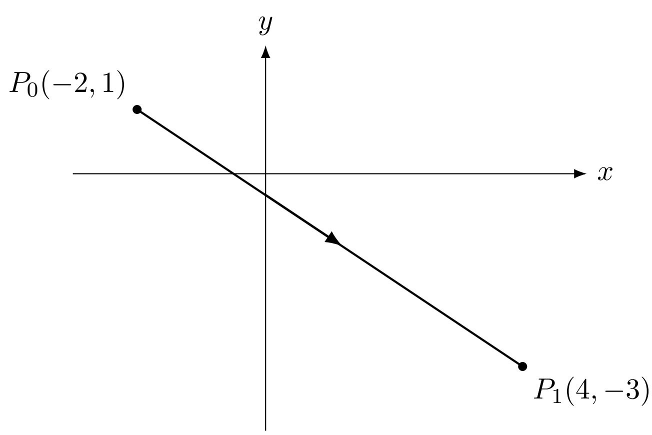
using $0 \leq t \leq 1$.

**Solution.** A standard line-segment parametrization is

$$\mathbf{r}(t) = (1-t)P_0 + tP_1, \qquad 0 \leq t \leq 1.$$

Thus

$$\mathbf{r}(t) = (1-t)(-2,1) + t(4,-3) = (-2+6t,\ 1-4t), \qquad 0 \leq t \leq 1.$$



7. Give parametric equations for the circle of radius 3 centered at $(2, -1)$ that

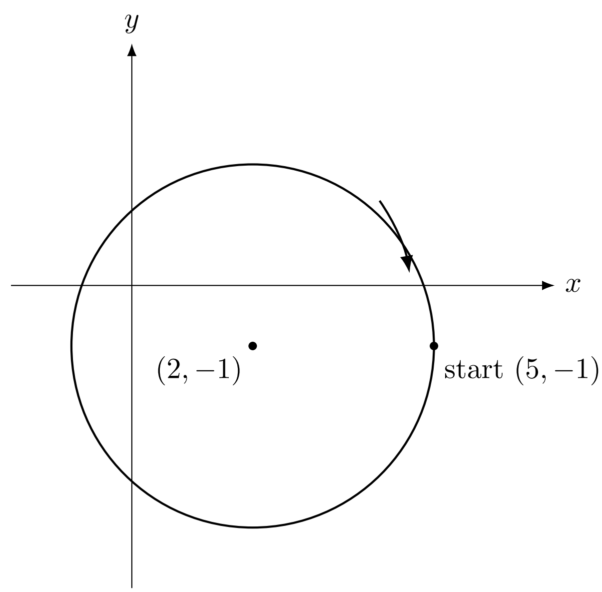
   - starts at the *rightmost* point of the circle, and
   - travels *clockwise* exactly once.

   **Solution.** A circle of radius 3 centered at $(2, -1)$ can be written

   $$x = 2 + 3\cos t, \qquad y = -1 + 3\sin t.$$

   This starts at the rightmost point $(5, -1)$ when $t = 0$ and goes counterclockwise. To make it go *clockwise*, negate the sine term:

   $$\boxed{x = 2 + 3\cos t, \qquad y = -1 - 3\sin t, \qquad 0 \le t \le 2\pi.}$$



8. Give a parametrization of the upper semicircle

$$x^2 + y^2 = 9, \qquad y \geq 0,$$

traced from $(-3, 0)$ to $(3, 0)$ (left to right). State an interval for $t$.

**Solution.** The circle $x^2 + y^2 = 9$ has radius 3. A standard parametrization is

$$x = 3\cos t, \qquad y = 3\sin t.$$

This traces the *upper* semicircle when $0 \leq t \leq \pi$, but it goes from $(3, 0)$ to $(-3, 0)$ (right to left). To go left to right, use

$$x = -3\cos t, \qquad y = 3\sin t.$$

Then

$$t = 0 : \ (-3, 0), \qquad t = \pi : \ (3, 0),$$

and $y \geq 0$ for $0 \leq t \leq \pi$. So

$$\boxed{x = -3\cos t, \qquad y = 3\sin t, \qquad 0 \leq t \leq \pi.}$$