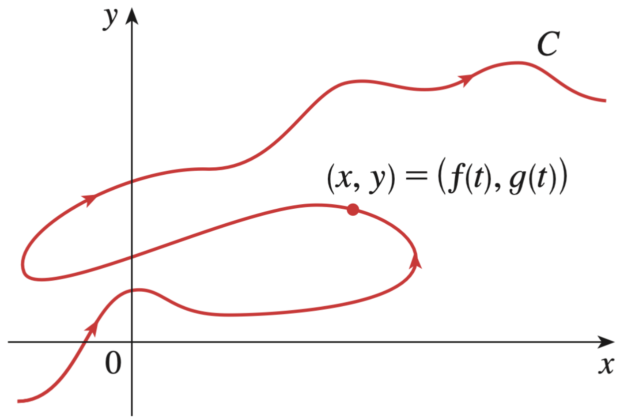
10.1 Curves Defined by Parametric Equations

Question. Imagine that a particle moves along the curve C shown below.



Is it possible to describe C by an equation of the form $y = f(x)$?

No. C fails the vertical line test.

Question. How can we describe the curve C?

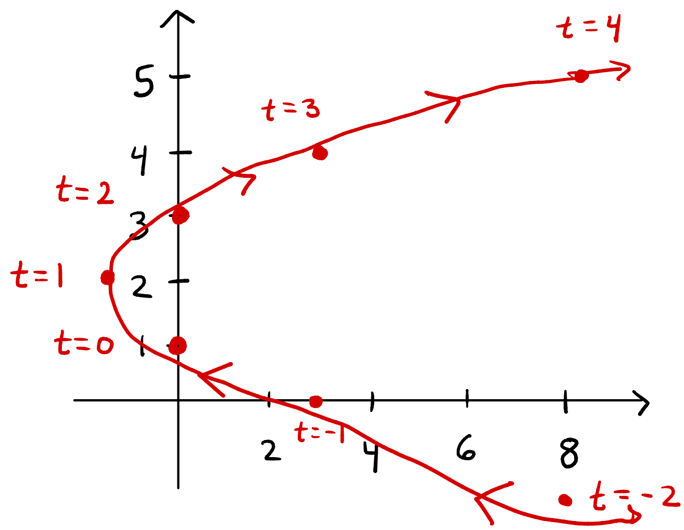
- Treat the x- and y-coordinates as functions of a third variable t, called the parameter
- Write down equations $x = f(t)$ and $y = g(t)$, called the parametric equations
- As t varies, the point $(x,y) = (f(t), g(t))$ traces out a curve, called a parametric curve.

Remark. The parameter t does not necessarily represent time. However, in many applications of parametric curves, t does denote time and we can interpret $(x, y) = (f(t), g(t))$ as the position of a moving object at time t.

Example. Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



- The marked points appear at equal time intervals, but not at equal distances

- From the second equation, $t = y - 1$

- Plugging this in to EQ#1, $x = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$

- Conclude: Every point lies on the parabola $x = y^2 - 4y + 3$ and since can be anything, we trace out the whole parabola.

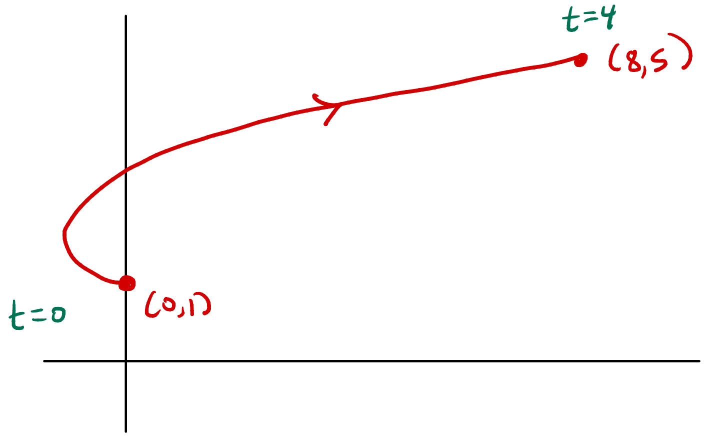
Remark. The process used in this example is called eliminating the parameter.

Remark. Eliminating the parameter can be helpful in identifying the shape of the parametric curve, but we lose some information in the process. The equation in x and y describes the curve the particle travels along, whereas the parametric equations could tell us where the particle is at any given time and indicate the direction of motion.

Remark. It is not always possible to eliminate the parameter from parametric equations. There are many parametric curves that don't have a representation as an equation in x and y.

Example. Sometimes we restrict t to lie in a particular interval. Graph the parametric curve given by

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \le t \le 4$$

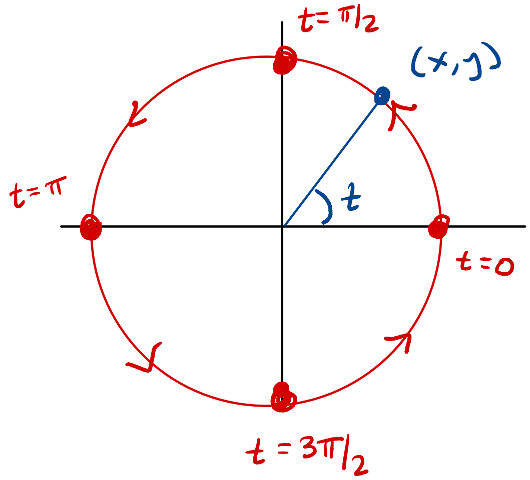


This is the same curve as before, but we only plot values between $t=0$ and $t=4$

Example. What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi$$

t	x	y
0	1	0
$\pi/2$	0	1
π	-1	0
$3\pi/2$	0	-1
2π	1	0

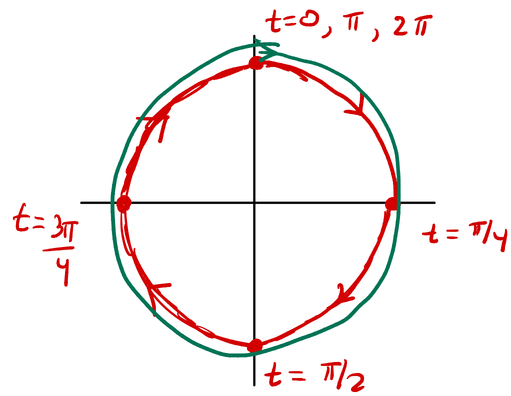


The parameter t is the angle (in radians) of the point (x,y).

Example. What curve is represented by the given parametric equations?

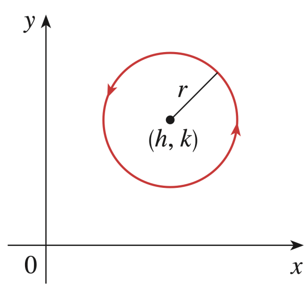
$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \le t \le 2\pi$$

t	x	y
0	0	1
$\pi/4$	1	0
$\pi/2$	0	-1
$3\pi/4$	-1	0
π	0	1
2π	0	1



This produces a circle starting at $(0,1)$, traveling twice clockwise around the circle

Example. Find parametric equations for the circle with center (h, k) and radius r.



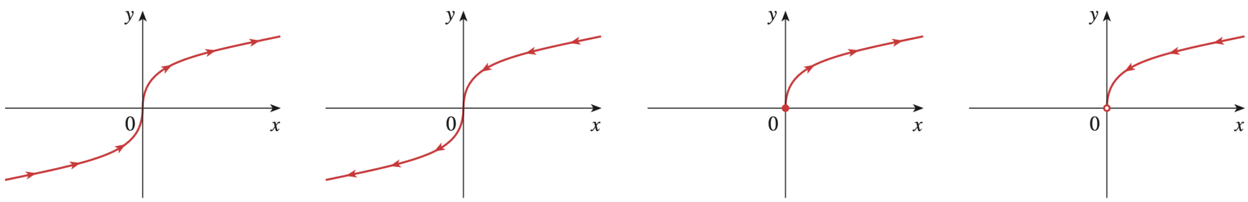
A circle with radius r centered at the origin (traced counterclockwise), is given by

$$x = r\cos t \qquad y = r\sin t$$

To move the center to (h,k) we can write

$$x = h + r\cos t \qquad y = k + r\sin t$$

Example. Each of the following sets of parametric equations gives the position of a moving particle at time t.



(a) $x = t^3, y = t$ (b) $x = -t^3, y = -t$ (c) $x = t^{3/2}, y = \sqrt{t}$ (d) $x = e^{-3t}, y = e^{-t}$

In each case, eliminating the parameter gives $x = y^3$, so each particle moves along the cubic curve $x = y^3$; however, the particles move in different ways

(a) the particle moves from left to right

(b) the particle moves from right to left

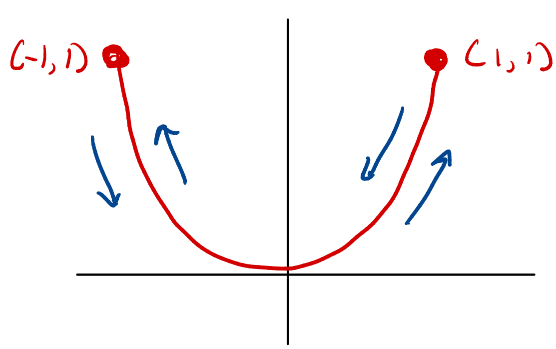
(c) The equations are only defined for $t \geq 0$

(d) Here $x > 0$, $y > 0$ and there is a hole at the origin.

Example. Sketch the curve with parametric equations $x = \sin t, y = \sin^2 t$.

$y = (\sin t)^2 = x^2$, so the point (x,y) moves on the parabola $y = x^2$

$-1 \leq \sin t \leq 1$, so $-1 \leq x \leq 1$. Hence we only trace some of the parabola

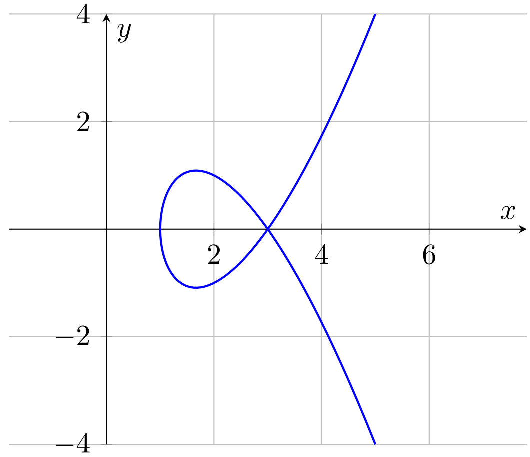


$\sin t$ is periodic, the point will oscillate back and forth along the curve

Example. Consider the curve represented by the parametric equations

$$x = t^2 + 1, \quad y = t^3 - 2t \quad \text{for} \quad -2 \le t \le 2.$$

Find the point at which the curve intersects itself and the corresponding values of t.



We need to solve the system

a and b are two different values of t

① $x(a) = x(b)$ ② $y(a) = y(b)$

where $a \neq b$.

① $a^2 + 1 = b^2 + 1$

$a^2 = b^2$

$\cancel{a = b}$ or $a = -b$

② $a^3 - 2a = b^3 - 2b$

$a^3 - b^3 = 2a - 2b$

$\cancel{(a-b)}(a^2 + ab + b^2) = 2\cancel{(a-b)}$

We can divide by $a-b$ since $a \neq b$

$a^2 + ab + b^2 = 2$

Substitute ① into ②, we get $b^2 + (-b)(b) + b^2 = 2 \Rightarrow b^2 = 2 \Rightarrow b = \pm\sqrt{2}$

Conclude: The two values of t are $\pm\sqrt{2}$

(if $b = \sqrt{2}$, then $a = -\sqrt{2}$
if $b = -\sqrt{2}$, then $a = \sqrt{2}$)

Substitute either $t=\sqrt{2}$ or $t=-\sqrt{2}$ into the parametric equations

$$x = (\sqrt{2})^2 + 1 = 2+1 = 3$$

$$y = (\sqrt{2})^3 - 2(\sqrt{2}) = 2\sqrt{2} - 2\sqrt{2} = 0$$

The curve intersects itself at $(3,0)$ when $t = \pm\sqrt{2}$