Quiz 1 Practice (Again!)

Evaluate each integral. Show all work.

1. Evaluate $\int e^{3x}\cos(2x)\,dx$.

2. Evaluate $\int \sin^5 x\cos^6 x\,dx$.

3. Evaluate $\int \frac{\sqrt{9x^2-4}}{x^2}\,dx$.

4. Evaluate $\int \frac{x^3}{(x^2+1)^2}\,dx$.

5. Evaluate $\int \frac{x^2+7x+1}{(x-1)^2(x+2)}\,dx$.

6. Evaluate $\int x\arctan x\,dx$.

1. Evaluate $\underbrace{\int e^{3x}\cos(2x)\,dx}_{I}$.

LIATE

Product → I.B.P. → Boomerang!

$u = \cos(2x)$ $\quad v = \frac{1}{3}e^{3x}$

$du = -2\sin(2x)dx$ $\quad dv = e^{3x}dx$

① $I = \underbrace{\cos(2x)\cdot\frac{1}{3}e^{3x}}_{u\cdot v} - \int \underbrace{\frac{1}{3}e^{3x}\cdot(-2\sin(2x))\,dx}_{v\cdot du}$

$= \frac{1}{3}e^{3x}\cos(2x) + \frac{2}{3}\underbrace{\int e^{3x}\sin(2x)\,dx}_{J}$

$u = \sin(2x)$ $\quad v = \frac{1}{3}e^{3x}$

$du = 2\cos(2x)dx$ $\quad dv = e^{3x}dx$

② $J = \underbrace{\sin(2x)\cdot\frac{1}{3}e^{3x}}_{u\cdot v} - \int \underbrace{\frac{1}{3}e^{3x}\cdot 2\cos(2x)dx}_{v\cdot du}$

$= \frac{1}{3}e^{3x}\sin(2x) - \frac{2}{3}\int e^{3x}\cos(2x)\,dx$

$= \frac{1}{3}e^{3x}\sin(2x) - \frac{2}{3}I$

③ In total,

$$I = \frac{1}{3}e^{3x}\cos(2x) + \frac{2}{3}\left(\frac{1}{3}e^{3x}\sin(2x) - \frac{2}{3}I\right)$$

$$I = \frac{1}{3}e^{3x}\cos(2x) + \frac{2}{9}e^{3x}\sin(2x) - \frac{4}{9}I$$

$$\frac{13}{9}I = \frac{1}{3}e^{3x}\cos(2x) + \frac{2}{9}e^{3x}\sin(2x)$$

$$I = \frac{9}{13}\left(\frac{1}{3}e^{3x}\cos(2x) + \frac{2}{9}e^{3x}\sin(2x)\right) + C$$

Speed Tips:

- Use I and J
- Don't worry about simplifying final answer.

Trig Powers $\to$ Trig Integral

2. Evaluate $\int \sin^5 x \cos^6 x \, dx$.

$u = \sin x$
$du = \cos x \, dx$

can't convert

$\int \sin^5 x \cdot \boxed{\cos^5 x} \cdot \cos x \, dx$

$u = \cos x$
$du = -\sin x \, dx$

$-\int \boxed{\sin^4 x} \cdot \cos^6 x \cdot (-\sin x) \, dx$

converts ✓

$$\int \sin^5 x \cos^6 x \, dx = -\int \sin^4 x \cdot \cos^6 x \cdot (-\sin x) \, dx$$

$$= -\int (\sin^2 x)^2 \cdot \cos^6 x \cdot (-\sin x) \, dx$$

$$= -\int (1-\cos^2 x)^2 \cdot \cos^6 x \cdot (-\sin x) \, dx$$

Let $u = \cos x$, $du = -\sin x \, dx$

$$= -\int (1-u^2)^2 \cdot u^6 \cdot du$$

$$= -\int (1-2u^2+u^4) \cdot u^6 \, du$$

$$= -\int u^6 - 2u^8 + u^{10} \, du$$

$$= -\frac{u^7}{7} + \frac{2u^9}{9} - \frac{u^{11}}{11} + C$$

$$= -\frac{\cos^7 x}{7} + \frac{2\cos^9 x}{9} - \frac{\cos^{11} x}{11} + C$$

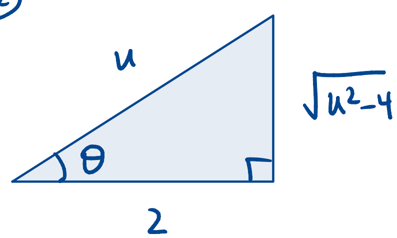
3. Evaluate $\displaystyle\int \frac{\sqrt{9x^2-4}}{x^2}\,dx$.

$\sqrt{\text{quadratic}} \rightsquigarrow$ Trig Sub

① Let $u = 3x$. Then $du = 3dx$

$$\int \frac{\sqrt{9x^2-4}}{x^2}\,dx = \int \frac{\sqrt{u^2-4}}{\left(\frac{u}{3}\right)^2} \cdot \frac{1}{3}du = 3\int \frac{\sqrt{u^2-4}}{u^2}\,du$$

②



③ From the triangle,

- $\cos\theta = \frac{2}{u} \Rightarrow u = \frac{2}{\cos\theta} = 2\sec\theta$
- $du = 2\sec\theta\tan\theta\,d\theta$
- $\tan\theta = \frac{\sqrt{u^2-4}}{2} \Rightarrow \sqrt{u^2-4} = 2\tan\theta$

④
$$3\int \frac{\sqrt{u^2-4}}{u^2}\,du = 3\int \frac{2\tan\theta}{4\sec^2\theta} \cdot 2\sec\theta\tan\theta\,d\theta$$

$$= 3\int \frac{\tan^2\theta}{\sec\theta}\,d\theta = 3\int \tan^2\theta\cos\theta\,d\theta$$

$$= 3\int \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos\theta\,d\theta = 3\int \frac{\sin^2\theta}{\cos\theta}\,d\theta$$

$$= 3\int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta = 3\int \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}\,d\theta$$

$$= 3\int \sec\theta\,d\theta - 3\int\cos\theta\,d\theta$$

I have $\int \sec\theta\,d\theta$ memorized

$$= 3\ln\left|\sec\theta + \tan\theta\right| - 3\sin\theta + C$$

Use the triangle

$$= 3\ln\left|\frac{u}{2} + \frac{\sqrt{u^2-4}}{2}\right| - 3\cdot\frac{\sqrt{u^2-4}}{u} + C$$

$$= 3\ln\left|\frac{3x}{2} + \frac{\sqrt{9x^2-4}}{2}\right| - 3\cdot\frac{\sqrt{9x-4}}{3x} + C$$

4. Evaluate $\displaystyle\int \frac{x^3}{(x^2+1)^2}\,dx.$

Let $u = x^2+1$. Then $du = 2x \cdot dx \Rightarrow x \cdot dx = \frac{1}{2} du$

$$\int \frac{x^3}{(x^2+1)^2}\,dx = \int \frac{x^2}{(x^2+1)^2} \cdot x\,dx$$

$$= \frac{1}{2}\int \frac{(u-1)}{u^2}\,du$$

$$= \frac{1}{2}\int \frac{u}{u^2} - \frac{1}{u^2}\,du$$

$$= \frac{1}{2}\int \frac{1}{u} - u^{-2}\,du$$

$$= \frac{1}{2}\left(\ln|u| + u^{-1} + C\right)$$

$$= \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\cdot\frac{1}{x^2+1} + C$$

<u>Trig Sub</u>:

$$\int \frac{x^3}{(x^2+1)^2}\,dx = \int \frac{x^3}{\left(\sqrt{x^2+1}\right)^4}\,dx$$

use $x = \tan\theta$

<u>Partial Fractions</u>:

$$\int \frac{x^3}{(x^2+1)^2}\,dx = \int \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}\,dx$$

Try all three!

5. Evaluate $\int \frac{x^2+7x+1}{(x-1)^2(x+2)}\,dx$.

Rational Function ⇝ Partial Fractions

$$\frac{x^2+7x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$x^2+7x+1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Set $x=1$: $9 = 3B \Rightarrow B=3$

Set $x=-2$: $4-14+1 = 9C \Rightarrow C=-1$

Set $x=0$: $1 = A(-1)(2) + 3(2) + (-1)(1)$

$1 = -2A + 6 - 1$

$-4 = -2A \Rightarrow A=2$

$$\int \frac{2}{x-1} + \frac{3}{(x-1)^2} - \frac{1}{(x+2)}\,dx$$

$$2\ln|x-1| - \frac{3}{x-1} - \ln|x+2| + C$$

LIATE

6. Evaluate $\int x \arctan x \, dx$.

Product $\rightsquigarrow$ I.B.P.

$u = \arctan(x) \qquad v = \frac{1}{2}x^2$

$du = \frac{1}{1+x^2}\,dx \qquad dv = x\,dx$

①

$$\int x \arctan(x)\,dx = \frac{1}{2}x^2 \arctan(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

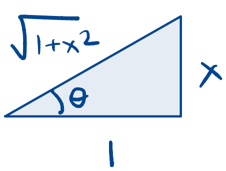
②

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2}{(\sqrt{1+x^2})^2}\,dx = \int \frac{\tan^2\theta}{\sec^2\theta}\sec^2\theta\,d\theta = \int \tan^2\theta\,d\theta$$

$$= \int \sec^2\theta - 1\,d\theta$$

$$= \tan\theta - \theta + C$$

$$= x - \arctan x + C$$



$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$\sqrt{1+x^2} = \sec\theta$

③

$$\int x \arctan x\,dx = \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}(x - \arctan x + C)$$

$$= \frac{1}{2}x^2 \arctan(x) + \frac{1}{2}\arctan x - \frac{1}{2}x + C$$

Alternate:

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{1+x^2}\,dx = \int 1 - \frac{1}{1+x^2}\,dx$$

$$= x - \arctan x + C$$