Quiz 1 Practice

Evaluate each integral. Show all work.

1. Evaluate $\displaystyle\int_1^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx.$

2. Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx.$

3. Evaluate $\displaystyle\int \frac{3x^2+3x+4}{x^3+4x}\,dx.$

4. Evaluate $\displaystyle\int \frac{\ln x}{\sqrt[3]{x}}\,dx.$

5. Evaluate $\displaystyle\int \tan^5 x \sec^4 x\,dx.$

6. Evaluate $\displaystyle\int e^{-2x}\cos(3x)\,dx.$

Function + Derivative → u-sub

1. Evaluate $\displaystyle\int_1^{\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$.

Let $u = \sqrt{x} = x^{1/2}$.

Then $du = \frac{1}{2} x^{-1/2} dx = \frac{1}{2\sqrt{x}} dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}} dx$

Bounds: When $x = 1$, $u = 1$

When $x = \pi^2$, $u = \pi$

$$\int_{x=1}^{x=\pi^2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = \int_{u=1}^{u=\pi} \sin(u) \cdot 2\,du$$

$$= \Big[-2\cos(u)\Big]_{u=1}^{u=\pi}$$

$$= -2\cos(\pi) + 2\cos(1)$$

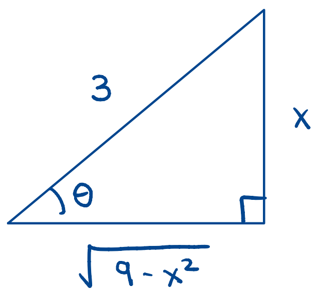
$$= 2 + 2\cos(1)$$

$\sqrt{\ }$ → Trig Sub

Know These →

Identity	Formula
Pythagorean	$\sin^2 x + \cos^2 x = 1$
Tangent Identity	$\tan^2 x + 1 = \sec^2 x$
Power-Reducing Formulas	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
Product-to-Sum	$\sin x \cos x = \frac{1}{2}\sin 2x$
Double-Angle Formulas	$\sin 2x = 2\sin x \cos x$, $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

2. Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9 - x^2}}\, dx$.

① 

② From the triangle,

- $\sin\theta = \frac{x}{3} \Rightarrow x = 3\sin\theta$
- $dx = 3\cos\theta\, d\theta$
- $\cos\theta = \frac{\sqrt{9-x^2}}{3} \Rightarrow \sqrt{9-x^2} = 3\cos\theta$

③
$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{(3\sin\theta)^2}{3\cos\theta}\cdot 3\cos\theta\, d\theta$$
$$= \int 9\sin^2\theta\, d\theta$$
> Trig identity
$$= \int 9\left(\frac{1-\cos(2\theta)}{2}\right) d\theta$$
$$= \frac{9}{2}\int 1 - \cos(2\theta)\, d\theta$$
$$= \frac{9}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C$$
$$= \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C$$
> Trig identity
$$= \frac{9}{2}\theta - \frac{9}{4}\cdot 2\sin\theta\cos\theta + C$$

④ From the triangle, $\theta = \sin^{-1}\left(\frac{x}{3}\right)$, $\sin\theta = \frac{x}{3}$, $\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$$
$$= \boxed{\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C}$$

Rational Function → Partial Fractions!

3. Evaluate $\int \frac{3x^2+3x+4}{x^3+4x}\,dx$.

① This is proper ✓

② Factor the denominator: $x^3+4x = x(x^2+4)$

③ Partial Fraction Decomposition:

$$\frac{3x^2+3x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

④ Find coefficients:

$$3x^2+3x+4 = A(x^2+4) + (Bx+C)x$$

$$3x^2+3x+4 = Ax^2+4A+Bx^2+Cx$$

$$3x^2+3x+4 = (A+B)x^2+Cx+4A$$

$$\begin{aligned} A+B&=3 \\ C&=3 \\ 4A&=4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} A&=1 \\ B&=2 \\ C&=3 \end{aligned}$$

⑤ Integrate:

$$\int \frac{3x^2+3x+4}{x^3+4x}\,dx = \int \frac{1}{x} + \frac{2x+3}{x^2+4}\,dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{2x}{x^2+4}\,dx + \int \frac{3}{x^2+4}\,dx$$

$$= \ln|x| + \ln|x^2+4| + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

* I have $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ memorized.

(Otherwise, factor out a^2 and use u-sub.)

Product → Integration by Parts

4. Evaluate $\int \frac{\ln x}{\sqrt[3]{x}}\,dx$.

LIATE

$$\int \frac{\ln(x)}{\sqrt[3]{x}}\,dx = \int x^{-1/3} \cdot \ln(x)\,dx$$

$u = \ln(x)$ $\qquad$ $v = \frac{3}{2}x^{2/3}$

$du = \frac{1}{x}\,dx$ $\qquad$ $dv = x^{-1/3}\,dx$

u · dv $\qquad$ u · v $\qquad$ v · du

$$\int x^{-1/3} \cdot \ln(x)\,dx = \ln(x) \cdot \frac{3}{2}x^{2/3} - \int \frac{3}{2}x^{2/3} \cdot \frac{1}{x}\,dx$$

$$= \ln(x) \cdot \frac{3}{2}x^{2/3} - \int \frac{3}{2}x^{-1/3}\,dx$$

$$= \ln(x) \cdot \frac{3}{2}x^{2/3} - \frac{9}{4}x^{2/3} + C$$

Trig Powers → Trig Integral!

5. Evaluate $\int \tan^5 x \sec^4 x \, dx$.

$u = \tan x$, $du = \sec^2 x \, dx$: $\int \tan^5 x \cdot \boxed{\sec^2 x} \cdot \underbrace{\sec^2 x \, dx}_{du}$ converts ✓

$u = \sec x$, $du = \sec x \tan x \, dx$: $\int \boxed{\tan^4 x} \cdot \sec^3 x \cdot \underbrace{\sec x \tan x \, dx}_{du}$ converts ✓

$$\int \tan^5 x \sec^4 x \, dx = \int \tan^5 x \cdot \sec^2 x \cdot \sec^2 x \, dx$$

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \Rightarrow 1 + \tan^2 x = \sec^2 x$$

$$= \int \tan^5 x \cdot (1 + \tan^2 x) \cdot \sec^2 x \, dx$$

Let $u = \tan x$, $du = \sec^2 x \, dx$

$$= \int u^5 \cdot (1 + u^2) \cdot du$$

$$= \int u^5 + u^7 \, du$$

$$= \frac{u^6}{6} + \frac{u^8}{8} + C$$

$$= \frac{\tan^6 x}{6} + \frac{\tan^8 x}{8} + C$$

Product → I.B.P → Boomerang!

LIATE

6. Evaluate $\int e^{-2x}\cos(3x)\,dx$.

$u = \cos(3x)$ $\quad v = -\frac{1}{2}e^{-2x}$

$du = -3\sin(3x)\,dx$ $\quad dv = e^{-2x}\,dx$

①

$$\int e^{-2x}\cos(3x)\,dx = -\frac{1}{2}e^{-2x}\cos(3x) - \int \left(-\frac{1}{2}e^{-2x}\right)\cdot\left(-3\sin(3x)\right)dx$$

$$= -\frac{1}{2}e^{-2x}\cos(3x) - \frac{3}{2}\int e^{-2x}\sin(3x)\,dx$$

$u = \sin(3x)$ $\quad v = -\frac{1}{2}e^{-2x}$

$du = 3\cos(3x)\,dx$ $\quad dv = e^{-2x}\,dx$

② Now,

$$\int e^{-2x}\sin(3x)\,dx = -\frac{1}{2}e^{-2x}\sin(3x) - \int \left(-\frac{1}{2}e^{-2x}\right)\cdot 3\cos(3x)\,dx$$

$$= -\frac{1}{2}e^{-2x}\sin(3x) + \frac{3}{2}\int e^{-2x}\cdot\cos(3x)\,dx$$

③ In total,

$$\int e^{-2x}\cos(3x)\,dx = -\frac{1}{2}e^{-2x}\cos(3x) - \frac{3}{2}\left(-\frac{1}{2}e^{-2x}\sin(3x) + \frac{3}{2}\int e^{-2x}\cos(3x)\,dx\right)$$

$$= -\frac{1}{2}e^{-2x}\cos(3x) + \frac{3}{4}e^{-2x}\sin(3x) - \frac{9}{4}\int e^{-2x}\cos(3x)\,dx$$

$$\Rightarrow \int e^{-2x}\cos(3x)\,dx = \frac{4}{13}\left(-\frac{1}{2}e^{-2x}\cos(3x) + \frac{3}{4}e^{-2x}\sin(3x)\right) + C$$