Applications of Integrals

A flowchart for choosing a setup based on the quantity being measured.

What am I computing?

AREA
Areas Between Curves

Use when: finding the area enclosed between two or more curves.

Vertical slices:

$$A = \int_a^b \Big[(\text{top}) - (\text{bottom})\Big]\,dx$$

Horizontal slices:

$$A = \int_c^d \Big[(\text{right}) - (\text{left})\Big]\,dy$$

VOLUME
Known Cross Sections

Use when: a solid has a known cross-sectional shape.

Formula: $V = \displaystyle\int_a^b A(x)\,dx$

VOLUME
Disks & Washers

Use when: rotating with slices perpendicular to the axis.

Formula: $V = \pi \displaystyle\int_a^b (R^2 - r^2)\,dx$

VOLUME
Cylindrical Shells

Use when: rotating with slices parallel to the axis.

Formula: $V = 2\pi \displaystyle\int (\text{radius})(\text{height})\,dx$

PHYSICAL APPLICATION
Work

Use when: computing work for springs, cables, and fluid-pumping problems.

Spring: $W = \displaystyle\int_a^b kx\,dx$

Cable: $W = \displaystyle\int_a^b \rho g\,(\text{distance})\,dx$

Tank: $W = \displaystyle\int_a^b \rho g\,A(x)\,(\text{distance})\,dx$

ACCUMULATION
Average Value

Use when: finding the mean height of a function on an interval.

Formula: $f_{\text{avg}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$

BALANCE
Center of Mass

Use when: finding the balancing point of point masses or a region under $y = f(x)$.

Discrete:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \qquad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}$$

Under $y = f(x)$ **with constant density:**

$$\bar{x} = \frac{1}{A}\int_a^b x\,f(x)\,dx$$

$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}\Big[f(x)\Big]^2\,dx$$

CURVES
Arc Length

Use when: finding the actual length of a curve.

Cartesian:

$$L = \int_a^b \sqrt{1 + \Big(f'(x)\Big)^2}\,dx$$

Parametric:

$$L = \int_a^b \sqrt{\Big(x'(t)\Big)^2 + \Big(y'(t)\Big)^2}\,dt$$

Polar:

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

POLAR
Polar Coordinates

Use when: finding the area of a region described by a polar curve $r = f(\theta)$.

Formula: $A = \dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$

PARAMETRIC
Parametric Calculus

Use when: a curve is described by $x = x(t)$ and $y = y(t)$.

Formula: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$