7.8 Improper Integrals

In defining a definite integral $\int_a^b f(x)\,dx$, we typically assume that the interval $[a,b]$ is finite and that f has no infinite discontinuities on $[a,b]$. In this section, we extend the notion of definite integrals to two situations:

1. The interval of integration is infinite (for example, from 1 to ∞)
2. The integrand has an infinite discontinuity at some point in the interval of integration.

In either case, the resulting integral is called an *improper integral*.

Definition (Improper Integrals of Type 1)**.** Let f be a function defined on $[a,\infty)$ or $(-\infty,b]$. We define the improper integral in the following ways:

(a) If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then

provided this limit exists (as a finite number). If the limit exists, we say the integral *converges*; otherwise, it *diverges*.

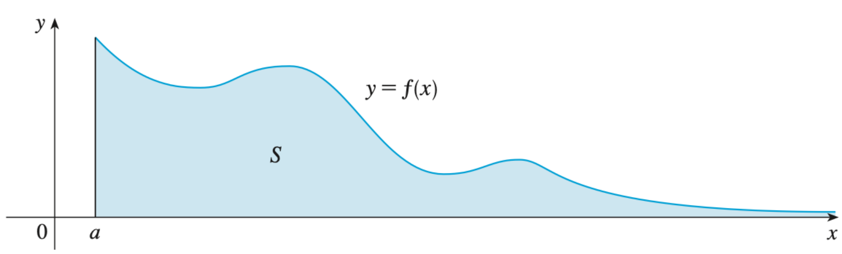
(b) If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, then

provided the limit exists (as a finite number).

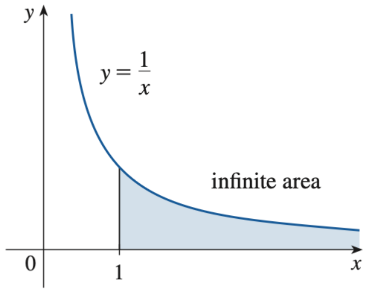
(c) If $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are each convergent, then

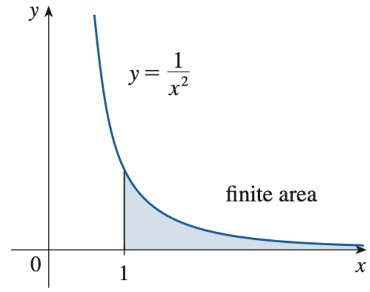
Any real number a can be used to split the interval.

Remark. These integrals can be interpreted as an area if f is a positive function.



Example. Show that $\int_1^\infty \frac{1}{x}$ is infinite, but $\int_1^\infty \frac{1}{x^2}$ is finite.





Example. For what values of p is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

Example. Evaluate $\int_{-\infty}^{0} xe^x \, dx$.

Type 2: Discontinuous Integrands

Sometimes the interval $[a, b]$ is finite, but the function f has a *vertical asymptote* or other kind of infinite discontinuity at some point in the interval. For instance, f might be continuous on $[a, b)$ and become unbounded as x approaches b.

Definition (Improper Integrals of Type 2)**.**

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

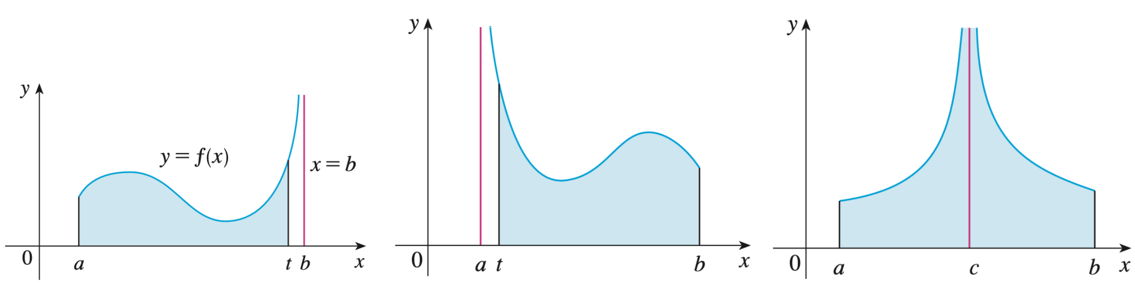
if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

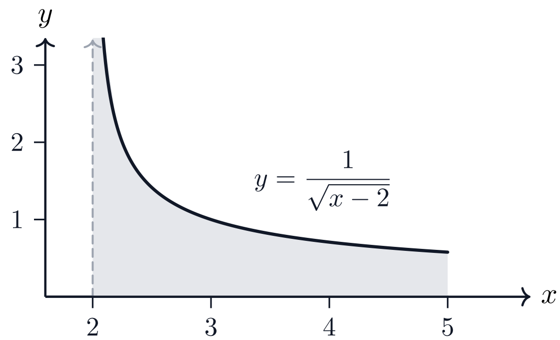
if this limit exists (as a finite number).

(c) If f has a discontinuity at some interior point c, where $a < c < b$, then we split:

provided both integrals on the right-hand side converge separately.



Example. Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.



Example. Evaluate $\displaystyle\int_0^3 \frac{dx}{x-1}$ if possible.

⚠ **Warning**. Had we ignored the vertical asymptote at $x = 1$, we might have incorrectly written:

$$\int_0^3 \frac{dx}{x-1} = \Big[\ln|x-1|\Big]_0^3 = \ln|2| - \ln|-1| = \ln 2,$$

Whenever you see $\int_a^b f(x)\,dx$, always check whether it is an improper integral!