

7.8 Improper Integrals: The Comparison Theorem

Sometimes we cannot compute an improper integral exactly, so we answer the next best question: does it converge or diverge? Instead of finding the value of $\int_a^\infty f(x) dx$, we ask whether the total area is finite or infinite. The Comparison Theorem helps by comparing $f(x)$ to a simpler function we already understand.

Theorem. Suppose f and g are continuous functions with $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

1. If $\int_a^\infty g(x) dx$ converges, then $\int_a^\infty f(x) dx$ also converges.
2. If $\int_a^\infty f(x) dx$ diverges, then $\int_a^\infty g(x) dx$ also diverges.

 **Warning.**

1. If the larger integral diverges, what can we conclude about the smaller integral?
2. If the smaller integral converges, what can we conclude about the larger integral?

The most common comparison integrals are p -integrals:

$$\int_1^\infty \frac{1}{x^p} dx \quad \begin{cases} \text{converges} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

Two especially useful examples are:

$$\int_1^\infty \frac{1}{x^2} dx \text{ converges} \quad \text{and} \quad \int_1^\infty \frac{1}{x} dx \text{ diverges.}$$

Another useful comparison integral is:

$$\int_1^\infty e^{-x} dx \text{ converges.}$$

Example. Determine whether $\int_1^{\infty} \frac{1 + \sin^4(2x)}{\sqrt{x}} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

Write the inequality:

The comparison integral is:

This comparison integral converges/diverges because:

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_2^{\infty} \frac{1}{x + e^x} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

Write the inequality:

The comparison integral is:

This comparison integral converges/diverges because:

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_2^{\infty} \frac{\cos^2 x}{x^2} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

Write the inequality:

The comparison integral is:

This comparison integral converges/diverges because:

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_1^{\infty} e^{-x^2} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

Write the inequality:

The comparison integral is:

This comparison integral converges/diverges because:

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_1^{\infty} \frac{3}{\sqrt{x^3 + x}} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

Write the inequality:

The comparison integral is:

This comparison integral converges/diverges because:

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_1^{\infty} \frac{5 + 2 \sin x}{x^2 + 2} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

Write the inequality:

The comparison integral is:

This comparison integral converges/diverges because:

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_3^{\infty} \frac{5 + 2 \sin x}{x - 2} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

Write the inequality:

The comparison integral is:

This comparison integral converges/diverges because:

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_3^{\infty} \frac{1}{\sqrt{x^2 - 1}} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

Write the inequality:

The comparison integral is:

This comparison integral converges/diverges because:

Therefore, the original integral:

converges

diverges