

7.8 Improper Integrals: The Comparison Theorem

Sometimes we cannot compute an improper integral exactly, so we answer the next best question: does it converge or diverge? Instead of finding the value of $\int_a^\infty f(x) dx$, we ask whether the total area is finite or infinite. The Comparison Theorem helps by comparing $f(x)$ to a simpler function we already understand.

Theorem. Suppose f and g are continuous functions with $0 \leq f(x) \leq g(x)$ for all $x \geq a$.

1. If $\int_a^\infty g(x) dx$ converges, then $\int_a^\infty f(x) dx$ also converges.
2. If $\int_a^\infty f(x) dx$ diverges, then $\int_a^\infty g(x) dx$ also diverges.

 **Warning.**

1. If the larger integral diverges, what can we conclude about the smaller integral?

Being smaller than ∞ means nothing ($\int_a^\infty f(x) dx$ could converge OR diverge).

2. If the smaller integral converges, what can we conclude about the larger integral?

Being larger than finite means nothing ($\int_a^\infty g(x) dx$ could converge OR diverge).

The most common comparison integrals are p -integrals:

$$\int_1^\infty \frac{1}{x^p} dx \quad \begin{cases} \text{converges} & \text{if } p > 1, \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

Two especially useful examples are:

$$\int_1^\infty \frac{1}{x^2} dx \text{ converges} \quad \text{and} \quad \int_1^\infty \frac{1}{x} dx \text{ diverges.}$$

Another useful comparison integral is:

$$\int_1^\infty e^{-x} dx \text{ converges.}$$

Example. Determine whether $\int_1^{\infty} \frac{1 + \sin^4(2x)}{\sqrt{x}} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

$$\frac{1}{\sqrt{x}}$$

Write the inequality:

bigger
numerator $\rightarrow \frac{1 + \sin^4(2x)}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} \geq 0$ on $[1, \infty)$ ✓

The comparison integral is:

$$\int_1^{\infty} \frac{1}{\sqrt{x}} dx$$

This comparison integral converges/diverges because:

$\int_1^{\infty} \frac{1}{\sqrt{x}} dx$ is a p-integral with $p = 1/2 \leq 1$

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_2^{\infty} \frac{1}{x+e^x} dx$ converges or diverges.

Guess:

converges

diverges

seems like $\frac{1}{e^x}$

A good comparison function is:

$$\frac{1}{e^x}$$

Write the inequality:

$$0 \leq \frac{1}{x+e^x} \leq \frac{1}{e^x} \quad \text{on } [2, \infty) \quad \checkmark$$

bigger denominator ↗

The comparison integral is:

$$\int_2^{\infty} \frac{1}{e^x} dx$$

This comparison integral converges/diverges because:

We know $\int_1^{\infty} \frac{1}{e^x} dx$ converges

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_2^{\infty} \frac{\cos^2 x}{x^2} dx$ converges or diverges.

Guess:

converges diverges

A good comparison function is:

$$\frac{1}{x^2}$$

Write the inequality:

$$0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2} \quad \text{on } [2, \infty) \quad \checkmark$$

smaller numerator

The comparison integral is:

$$\int_2^{\infty} \frac{1}{x^2} dx$$

This comparison integral converges/diverges because:

$$\int_2^{\infty} \frac{1}{x^2} dx \text{ is a } p\text{-integral with } p=2 > 1$$

Therefore, the original integral:

converges diverges

Example. Determine whether $\int_1^{\infty} e^{-x^2} dx$ converges or diverges.

Guess:

converges diverges

A good comparison function is:

$$\frac{1}{e^x}$$

Write the inequality:

$$0 \leq \frac{1}{e^{x^2}} \leq \frac{1}{e^x} \quad \text{on } [1, \infty) \quad \checkmark$$

bigger denominator →

The comparison integral is:

$$\int_1^{\infty} \frac{1}{e^x} dx$$

This comparison integral converges/diverges because:

We know $\int_1^{\infty} \frac{1}{e^x} dx$ converges.

Therefore, the original integral:

converges diverges

Example. Determine whether $\int_1^{\infty} \frac{3}{\sqrt{x^3+x}} dx$ converges or diverges.

Guess:

converges diverges

A good comparison function is:

$$\frac{3}{x^{3/2}}$$

Write the inequality:

$$0 \leq \frac{3}{\sqrt{x^3+x}} \leq \frac{3}{x^{3/2}} \quad \text{on } [1, \infty)$$

bigger denominator ↗

The comparison integral is:

$$\int_1^{\infty} \frac{3}{x^{3/2}} dx$$

This comparison integral converges/diverges because:

$$\int_1^{\infty} \frac{3}{x^{3/2}} dx = 3 \int_1^{\infty} \frac{1}{x^{3/2}} dx \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^{3/2}} dx \text{ is a } p\text{-integral with } p = \frac{3}{2} > 1$$

Therefore, the original integral:

converges diverges

Example. Determine whether $\int_1^{\infty} \frac{5 + 2 \sin x}{x^2 + 2} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

$$\frac{7}{x^2}$$

Write the inequality:

$$0 \leq \frac{5 + 2 \sin x}{x^2 + 2} \leq \frac{7}{x^2} \quad \text{on } [1, \infty)$$

bigger numerator (pointing to 7)
smaller denominator (pointing to x^2)

The comparison integral is:

$$\int_1^{\infty} \frac{7}{x^2} dx$$

This comparison integral ~~converges~~ diverges because:

$$\int_1^{\infty} \frac{7}{x^2} dx = 7 \int_1^{\infty} \frac{1}{x^2} dx \quad \text{and} \quad \int_1^{\infty} \frac{1}{x^2} dx \quad \text{is a } p\text{-integral with } p=2 > 1$$

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_3^{\infty} \frac{5 + 2 \sin x}{x - 2} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

$$\frac{3}{x}$$

Write the inequality:

$$\frac{5 + 2 \sin x}{x - 2} \geq \frac{3}{x} \geq 0 \quad \text{on } [3, \infty)$$

Smaller numerator (pointing to the numerator of the comparison function)
bigger denominator (pointing to the denominator of the comparison function)

The comparison integral is:

$$\int_3^{\infty} \frac{3}{x} dx$$

This comparison integral converges diverges because:

$$\int_3^{\infty} \frac{3}{x} dx = 3 \int_3^{\infty} \frac{1}{x} dx \quad \text{and} \quad \int_3^{\infty} \frac{1}{x} dx \text{ is a } p\text{-integral with } p=1 \leq 1$$

Therefore, the original integral:

converges

diverges

Example. Determine whether $\int_3^{\infty} \frac{1}{\sqrt{x^2-1}} dx$ converges or diverges.

Guess:

converges

diverges

A good comparison function is:

$$\frac{1}{x}$$

Write the inequality:

$$\frac{1}{\sqrt{x^2-1}} \geq \frac{1}{\sqrt{x^2}} = \frac{1}{x} \geq 0 \quad \text{on } [3, \infty) \quad \checkmark$$

↑ bigger denominator

The comparison integral is:

$$\int_3^{\infty} \frac{1}{x} dx$$

This comparison integral converges/diverges because:

p-integral with $p=1 \leq 1$

Therefore, the original integral:

converges

diverges