

7.8 Improper Integrals

In defining a definite integral $\int_a^b f(x) dx$, we typically assume that the interval $[a, b]$ is finite and that f has no infinite discontinuities on $[a, b]$. In this section, we extend the notion of definite integrals to two situations:

1. The interval of integration is infinite (for example, from 1 to ∞)
2. The integrand has an infinite discontinuity at some point in the interval of integration.

In either case, the resulting integral is called an *improper integral*.

Definition (Improper Integrals of Type 1). Let f be a function defined on $[a, \infty)$ or $(-\infty, b]$. We define the improper integral in the following ways:

- (a) If $\int_a^t f(x) dx$ exists for every $t \geq a$, then

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

provided this limit exists (as a finite number). If the limit exists, we say the integral *converges*; otherwise, it *diverges*.

- (b) If $\int_t^b f(x) dx$ exists for every $t \leq b$, then

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

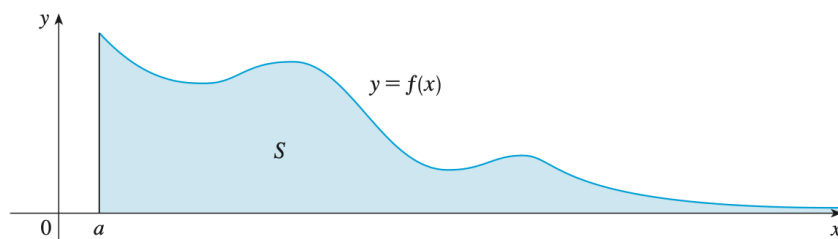
provided the limit exists (as a finite number).

- (c) If $\int_{-\infty}^a f(x) dx$ and $\int_a^{\infty} f(x) dx$ are each convergent, then

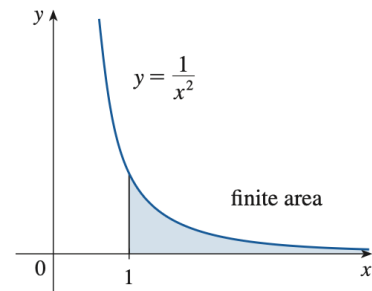
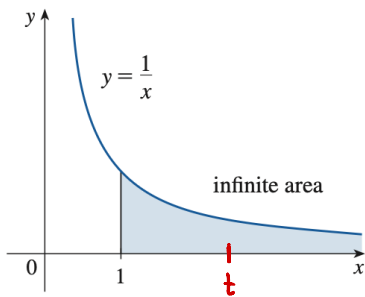
$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx$$

Any real number a can be used to split the interval.

Remark. These integrals can be interpreted as an area if f is a positive function.



Example. Show that $\int_1^{\infty} \frac{1}{x}$ is infinite, but $\int_1^{\infty} \frac{1}{x^2}$ is finite.



$$\begin{aligned} \int_1^{\infty} \frac{1}{x} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx \\ &= \lim_{t \rightarrow \infty} \left[\ln|x| \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left[\ln|t| - \ln|1| \right] \\ &= \lim_{t \rightarrow \infty} \ln t \end{aligned}$$

As $t \rightarrow \infty$, $\ln t \rightarrow \infty$

Diverges

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} dx \\ &= \lim_{t \rightarrow \infty} \left[-\frac{1}{x} \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left[-\frac{1}{t} - (-1) \right] \\ &= \lim_{t \rightarrow \infty} \left[1 - \frac{1}{t} \right] \end{aligned}$$

As $t \rightarrow \infty$, $\frac{1}{t} \rightarrow 0$

Converges to 1

Example. For what values of p is the integral $\int_1^{\infty} \frac{1}{x^p} dx$ convergent?

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \lim_{t \rightarrow \infty} \int_1^t x^{-p} dx \\ &= \lim_{t \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^t \quad > p \neq 1 \\ &= \lim_{t \rightarrow \infty} \left[\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1} \right] \\ &= \lim_{t \rightarrow \infty} \frac{1}{1-p} \left[t^{1-p} - 1 \right] \end{aligned}$$

If $p > 1$:

$1-p < 0$, so $t^{1-p} \rightarrow 0$

The limit converges

If $p < 1$:

$1-p > 0$, so $t^{1-p} \rightarrow \infty$

The limit diverges

* key result for the Comparison Theorem² & also for Series (p-test)

Example. Evaluate $\int_{-\infty}^0 x e^x dx$.

$$\text{We have } \int_{-\infty}^0 x e^x dx = \lim_{t \rightarrow -\infty} \int_t^0 x e^x dx$$

$$\begin{aligned} u &= x & v &= e^x \\ du &= dx & dv &= e^x dx \end{aligned}$$

$$\begin{aligned} \int_t^0 x e^x dx &= \left[x e^x \right]_t^0 - \int_t^0 e^x dx \\ &= (0 \cdot e^0 - t \cdot e^t) - [e^x]_t^0 \\ &= -t e^t - (e^0 - e^t) \\ &= -t e^t - 1 + e^t \end{aligned}$$

$$\text{Now, } \lim_{t \rightarrow -\infty} [-t e^t - 1 + e^t] = -1 - \lim_{t \rightarrow -\infty} t e^t + \lim_{t \rightarrow -\infty} e^t$$

indeterminate form
 $-\infty \cdot 0$

$$\lim_{t \rightarrow -\infty} t e^t = \lim_{t \rightarrow -\infty} \frac{t}{e^{-t}}$$

> L'Hospital type $\frac{\infty}{\infty}$

$$= \lim_{t \rightarrow -\infty} \frac{1}{-e^{-t}}$$

$$= \lim_{t \rightarrow -\infty} -e^t$$

$$= 0$$

$$\text{In total, we have } -1 - 0 + 0 = \boxed{-1}$$

Type 2: Discontinuous Integrands

Sometimes the interval $[a, b]$ is finite, but the function f has a *vertical asymptote* or other kind of infinite discontinuity at some point in the interval. For instance, f might be continuous on $[a, b)$ and become unbounded as x approaches b .

Definition (Improper Integrals of Type 2).

(a) If f is continuous on $[a, b)$ and is discontinuous at b , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a , then

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

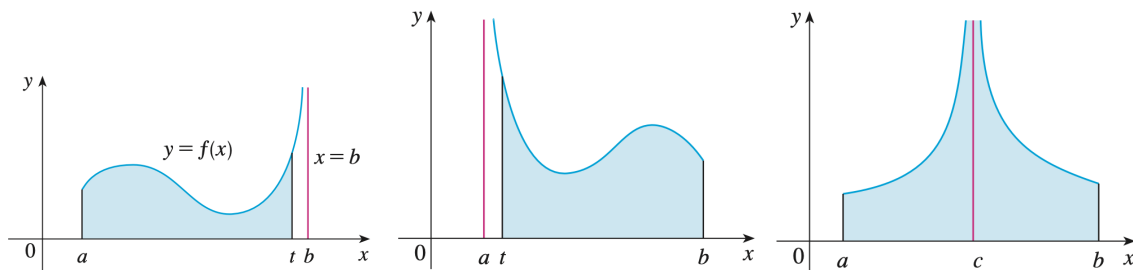
if this limit exists (as a finite number).

(c) If f has a discontinuity at some interior point c , where $a < c < b$, then we split:

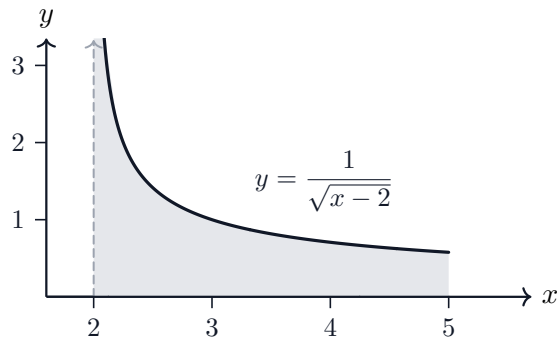
$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

← If one of these diverges, the integral diverges

provided both integrals on the right-hand side converge separately.



Example. Find $\int_2^5 \frac{1}{\sqrt{x-2}} dx$.



This is improper at $x=2$:

$$\int_2^5 \frac{1}{\sqrt{x-2}} dx = \lim_{t \rightarrow 2^+} \int_t^5 \frac{1}{\sqrt{x-2}} dx$$

① Focus on $\int_t^5 \frac{1}{\sqrt{x-2}} dx$ first:

Let $u = x-2$. Then $du = dx$.

$$\begin{aligned} \Rightarrow \int_{t-2}^3 \frac{1}{\sqrt{u}} du &= \left[2u^{1/2} \right]_{t-2}^3 \\ &= 2\sqrt{3} - 2\sqrt{t-2} \end{aligned}$$

$$\begin{aligned} \textcircled{2} \lim_{t \rightarrow 2^+} \left[2\sqrt{3} - 2\sqrt{t-2} \right] &= 2\sqrt{3} - 0 \\ &= \boxed{2\sqrt{3}} \end{aligned}$$

Example. Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

This is improper because $\frac{1}{x-1}$ has an asymptote at $x=1$, which is in $[0,3]$. Split the integral at $x=1$:

$$\int_0^3 \frac{1}{x-1} dx = \lim_{t \rightarrow 1^-} \int_0^t \frac{1}{x-1} dx + \lim_{r \rightarrow 1^+} \int_r^3 \frac{1}{x-1} dx$$

$$\textcircled{1} = \lim_{t \rightarrow 1^-} \left[\ln|x-1| \right]_0^t = \lim_{t \rightarrow 1^-} \left[\ln|t-1| - \ln| \cancel{-1} | \right]$$

$$= \lim_{t \rightarrow 1^-} \ln|t-1|$$

As $t \rightarrow 1^-$, $|t-1| \rightarrow 0^+ <$

$$= \lim_{u \rightarrow 0^+} \ln(u)$$

$$= -\infty$$

Since one of $\textcircled{1}$ and $\textcircled{2}$ diverges, the integral diverges.

⚠ Warning. Had we ignored the vertical asymptote at $x=1$, we might have incorrectly written:

$$\int_0^3 \frac{dx}{x-1} = \left[\ln|x-1| \right]_0^3 = \ln|2| - \ln|-1| = \ln 2,$$

Whenever you see $\int_a^b f(x) dx$, always check whether it is an improper integral!