7.7 Approximate Integration

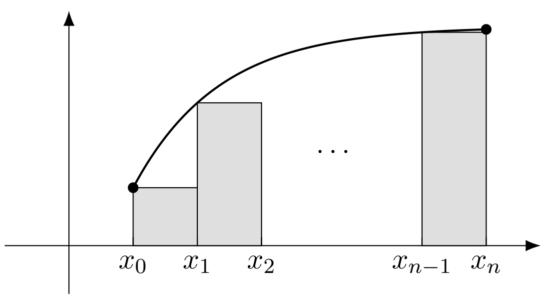
When we cannot find an exact antiderivative, or when we only have data from a graph or table, we estimate

$$\int_a^b f(x)\,dx$$

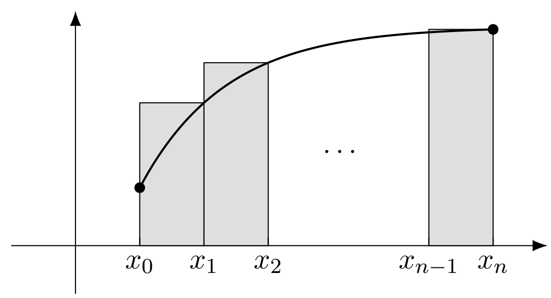
by adding up simple geometric areas. For each method, the subintervals are uniform. That is, $a = x_0$, $b = x_n$, and

$$\Delta x = \frac{b-a}{n}.$$

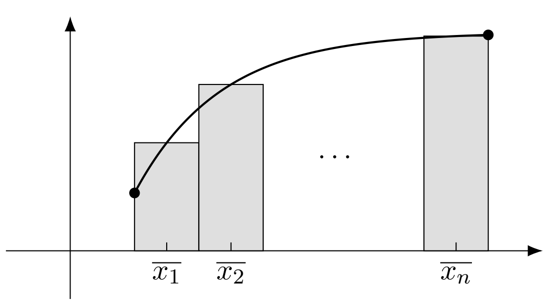
Left–endpoint approximation



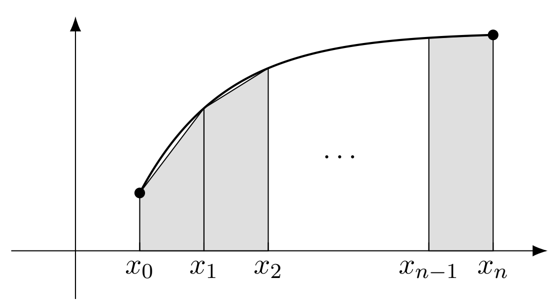
Right–endpoint approximation



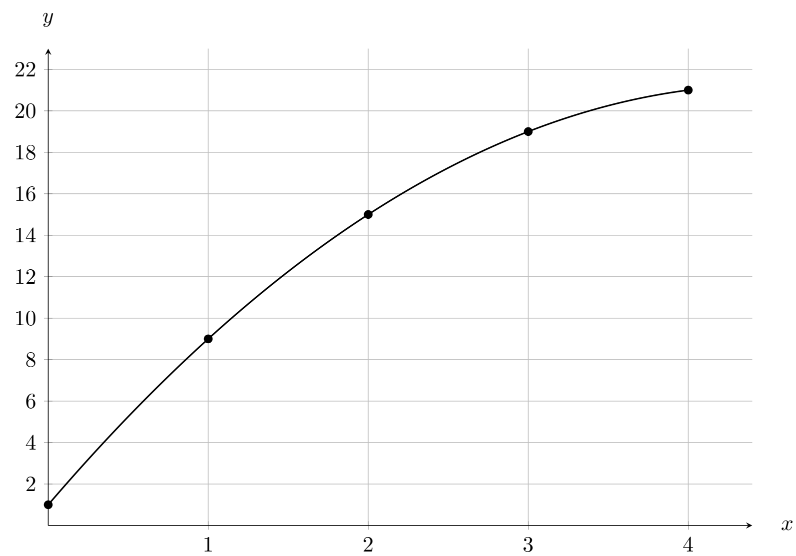
Midpoint approximation



Trapezoidal approximation

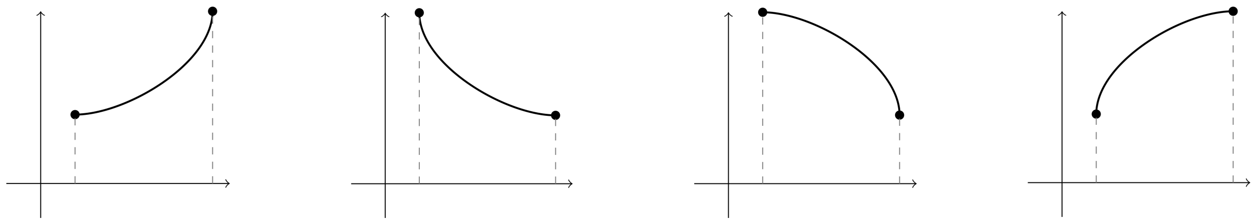


Example. Let $I = \int_0^4 f(x)\,dx$, where f is the increasing, concave down function shown below.



Find L_2, R_2, M_2, and T_2.

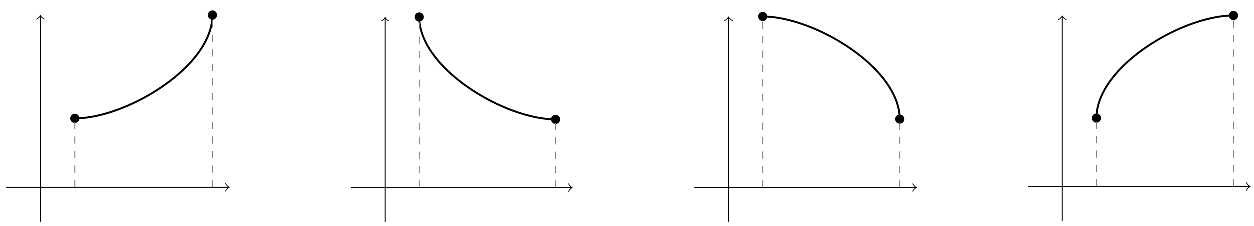
Example. L_n, for $n = 2$.



When $f(x)$ is ______________________, L_n is an overestimate.

When $f(x)$ is ______________________, L_n is an underestimate.

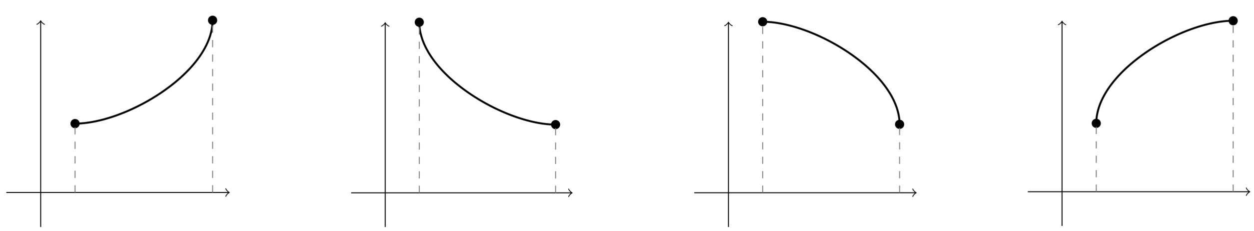
Example. R_n, for $n = 2$.



When $f(x)$ is ______________________, R_n is an overestimate.

When $f(x)$ is ______________________, R_n is an underestimate.

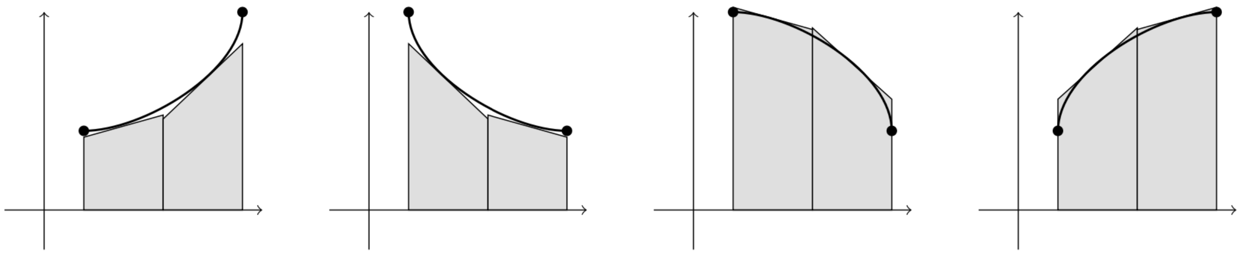
Example. T_n, for $n = 2$.



When $f(x)$ is ______________________, T_n is an overestimate.

When $f(x)$ is ______________________, T_n is an underestimate.

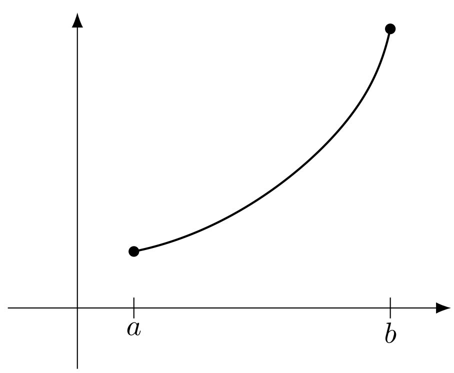
Example. M_n, with $n = 2$. By rotating the top of the rectangles of a Midpoint approximation, we can draw them as trapezoids.



When $f(x)$ is ____________________, M_n is an overestimate.

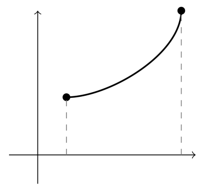
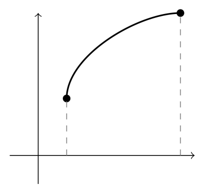
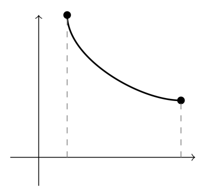
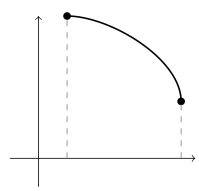
When $f(x)$ is ____________________, M_n is an underestimate.

Example. For $f(x)$ shown below, put L_n, R_n, M_n, T_n and $\int_a^b f(x)\,dx$ in order from smallest to largest.



__________ < __________ < __________ < __________ < __________

Summary of Orderings

Case	**Graph**	**Order**
Increasing, Concave Up		$L_n < M_n < \int f(x)dx < T_n < R_n$
Increasing, Concave Down		$L_n < T_n < \int f(x)dx < M_n < R_n$
Decreasing, Concave Up		$R_n < M_n < \int f(x)dx < T_n < L_n$
Decreasing, Concave Down		$R_n < T_n < \int f(x)dx < M_n < L_n$

1. Use increasing/decreasing to order L_n and R_n.

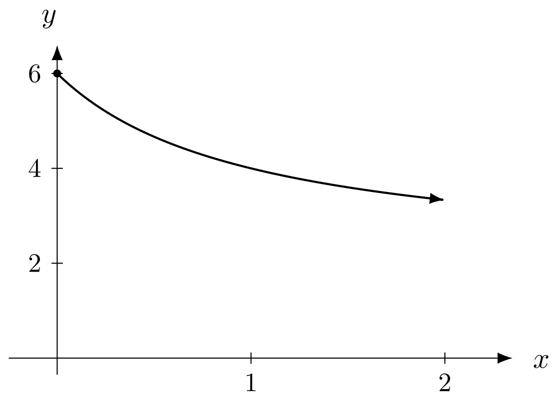
$$\begin{aligned} f \text{ increasing} &\implies L_n < R_n, \\ f \text{ decreasing} &\implies R_n < L_n. \end{aligned}$$

2. Use concavity to place the integral between M_n and T_n.

$$\begin{aligned} f \text{ concave up} &\implies M_n < \int_a^b f(x)\,dx < T_n, \\ f \text{ concave down} &\implies T_n < \int_a^b f(x)\,dx < M_n. \end{aligned}$$

Since M_n and T_n are both between L_n and R_n, combine the inequalities to order all five quantities from smallest to largest.

Example. Suppose f is decreasing and concave up on $[0, 2]$. Four approximations were used to estimate $\int_0^2 f(x)\,dx$. The estimates were $6, 8, 9$, and 12, and the same value of n was used for all four rules.



Which estimate came from L_n, R_n, M_n, and T_n? Between which two approximations does the true value of the integral lie?

Theorem. Suppose $|f''(x)| \le k$ for $a \le x \le b$. If E_T and E_M are the errors in the trapezoidal and midpoint approximations, then

$$|E_T| \le \frac{k(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \le \frac{k(b-a)^3}{24n^2}$$

Example. If we use the trapezoidal approximation with $n = 10$ to estimate $\int_1^3 x^3\,dx$, how accurate are we guaranteed to be?

Example. If we use the midpoint approximation with $n = 20$ to estimate $\int_0^1 \sin(2x)\,dx$, how accurate are we guaranteed to be?

Example. How large should n be to guarantee that using T_n to estimate $\int_0^1 e^{-3x}\,dx$ gives an error no larger than .001?

Example. Consider the following table of values.

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	2	3	4	6	7	9	10	10	11

(a) Use the midpoint rule with $n = 4$ to estimate $\int_0^4 f(x)\,dx$.

(b) Suppose that $-2 \leq f''(x) \leq 3$ for all x in $[0, 4]$. Use the midpoint error bound to estimate the error.