7.7 Approximate Integration

When we cannot find an exact antiderivative, or when we only have data from a graph or table, we estimate

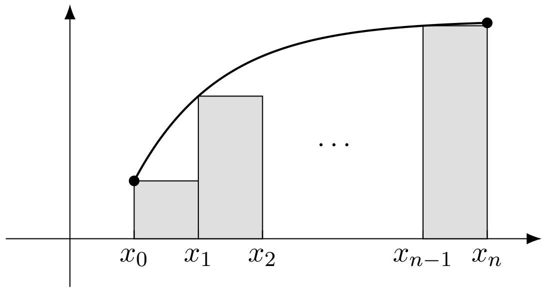
$$\int_a^b f(x)\,dx$$

by adding up simple geometric areas. For each method, the subintervals are uniform. That is, $a = x_0$, $b = x_n$, and

$$\Delta x = \frac{b-a}{n}.$$

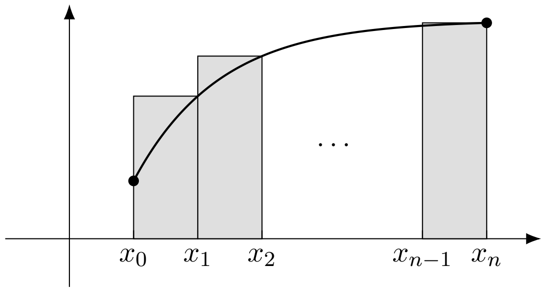
Left–endpoint approximation

$\Delta x = \frac{b-a}{n}$



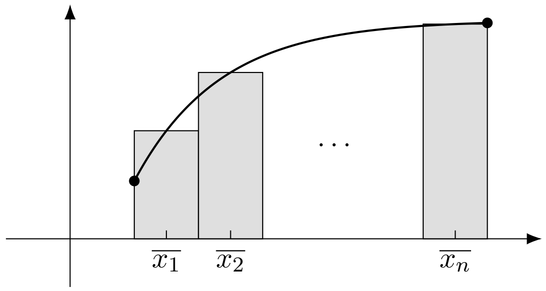
$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$
$$= \Delta x\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right]$$

Right–endpoint approximation



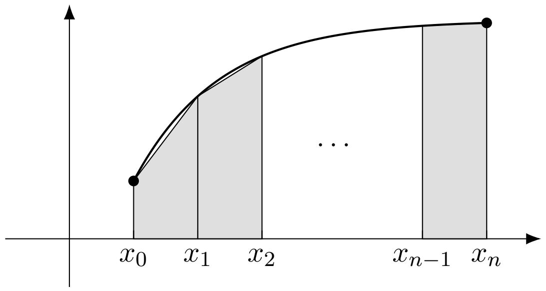
$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$
$$= \Delta x\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$$

Midpoint approximation



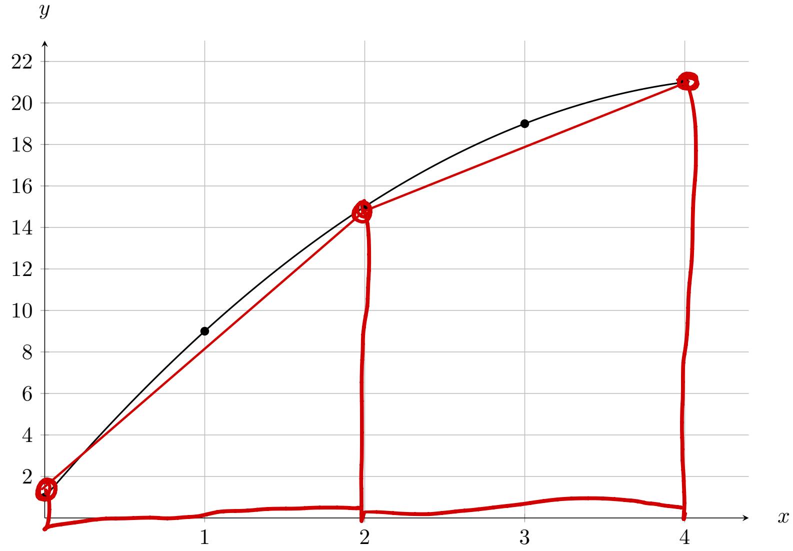
$$M_n = f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + \cdots + f(\bar{x}_n)\Delta x$$
$$= \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right]$$

Trapezoidal approximation



$$T_n = \frac{f(x_0)+f(x_1)}{2}\cdot\Delta x + \frac{f(x_1)+f(x_2)}{2}\cdot\Delta x + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\Delta x$$
$$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$
$$= \frac{1}{2}\left[L_n + R_n\right]$$

Example. Let $I = \int_0^4 f(x)\,dx$, where f is the increasing, concave down function shown below.



T_2

Find L_2, R_2, M_2, and T_2.

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{2} = 2$$

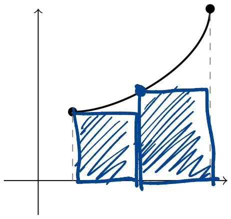
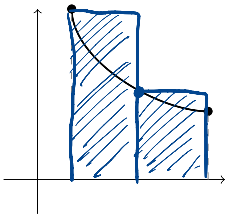
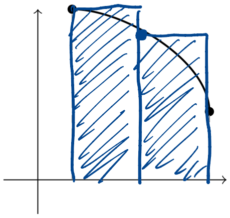
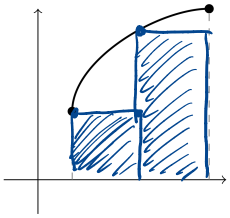
$$L_2 = 2 \cdot (1 + 15) = 32$$

$$R_2 = 2 \cdot (15 + 21) = 72$$

$$M_2 = 2 \cdot (9 + 19) = 56$$

$$T_2 = \frac{1}{2}(L_2 + R_2) = \frac{1}{2}(32 + 72) = 52$$

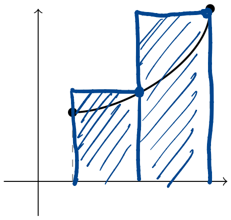
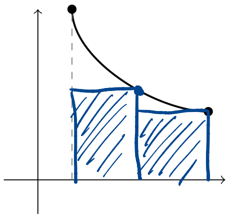
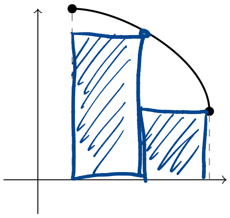
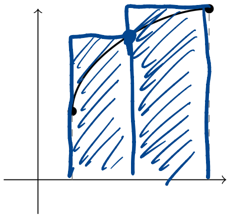
Example. L_n, for $n = 2$.

When $f(x)$ is decreasing, L_n is an overestimate.

When $f(x)$ is increasing, L_n is an underestimate.

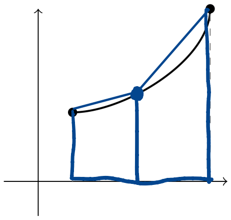
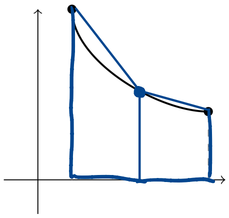
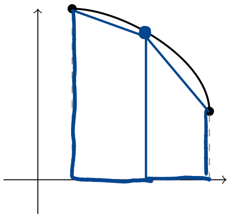
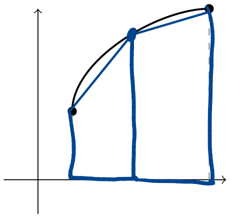
Example. R_n, for $n = 2$.

When $f(x)$ is increasing, R_n is an overestimate.

When $f(x)$ is decreasing, R_n is an underestimate.

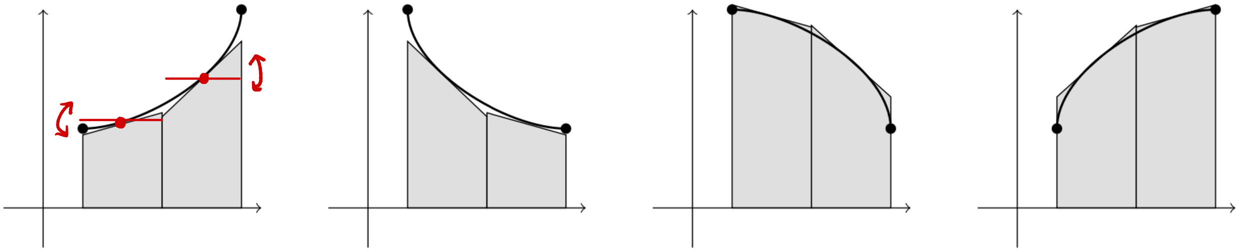
Example. T_n, for $n = 2$.

When $f(x)$ is concave up, T_n is an overestimate.

When $f(x)$ is concave down, T_n is an underestimate.

Example. M_n, with $n = 2$. By rotating the top of the rectangles of a Midpoint approximation, we can draw them as trapezoids.



When $f(x)$ is Concave down, M_n is an overestimate.

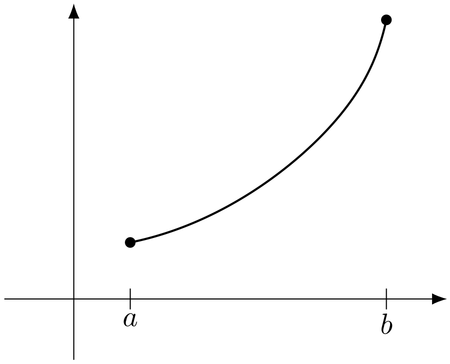
When $f(x)$ is Concave up, M_n is an underestimate.

Think about where the tangent line is

Example. For $f(x)$ shown below, put L_n, R_n, M_n, T_n and $\int_a^b f(x)\,dx$ in order from smallest to largest.

Key observation:

M_n, T_n, and $\int_a^b f(x)\,dx$ are always between L_n and R_n

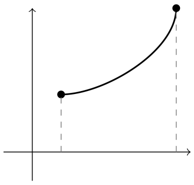
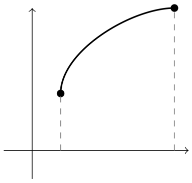
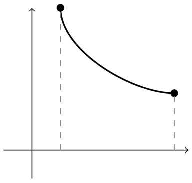
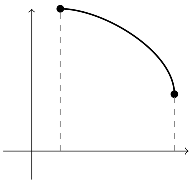


L_n < M_n < $\int_a^b f(x)\,dx$ < T_n < R_n

① Since f is increasing, $L_n < \int_a^b f(x)\,dx < R_n$

② Since f is concave up, $M_n < \int_a^b f(x)\,dx < T_n$

Summary of Orderings

Case	**Graph**	**Order**
Increasing, Concave Up		$L_n < M_n < \int f(x)dx < T_n < R_n$
Increasing, Concave Down		$L_n < T_n < \int f(x)dx < M_n < R_n$
Decreasing, Concave Up		$R_n < M_n < \int f(x)dx < T_n < L_n$
Decreasing, Concave Down		$R_n < T_n < \int f(x)dx < M_n < L_n$

1. Use increasing/decreasing to order L_n and R_n.

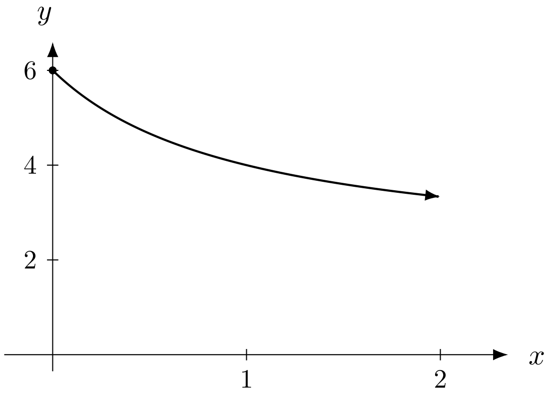
$$\begin{aligned} f \text{ increasing} &\implies L_n < R_n, \\ f \text{ decreasing} &\implies R_n < L_n. \end{aligned}$$

2. Use concavity to place the integral between M_n and T_n.

$$\begin{aligned} f \text{ concave up} &\implies M_n < \int_a^b f(x)\,dx < T_n, \\ f \text{ concave down} &\implies T_n < \int_a^b f(x)\,dx < M_n. \end{aligned}$$

Since M_n and T_n are both between L_n and R_n, combine the inequalities to order all five quantities from smallest to largest.

Example. Suppose f is decreasing and concave up on $[0,2]$. Four approximations were used to estimate $\int_0^2 f(x)\,dx$. The estimates were $6, 8, 9$, and 12, and the same value of n was used for all four rules.



Which estimate came from L_n, R_n, M_n, and T_n? Between which two approximations does the true value of the integral lie?

Decreasing : $R_n < L_n$

Concave up : $M_n < T_n$

$$\begin{array}{ccccccccc} R_n & < & M_n & < & \int_0^2 f(x)\,dx & < & T_n & < & L_n \\ \| & & \| & & & & \| & & \| \\ 6 & & 8 & & & & 9 & & 12 \end{array}$$

Therefore, $8 < \int_0^2 f(x)\,dx < 9$.

Theorem. Suppose $|f''(x)| \le k$ for $a \le x \le b$. If E_T and E_M are the errors in the trapezoidal and midpoint approximations, then

$$|E_T| \le \frac{k(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \le \frac{k(b-a)^3}{24n^2}$$

Example. If we use the trapezoidal approximation with $n = 10$ to estimate $\int_1^3 x^3\,dx$, how accurate are we guaranteed to be?

- $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$
- On $[1,3]$, $|f''(x)| \le 18$ since $f''(x)$ is increasing (max. at right endpoint)
- So, $|E_T| \le \dfrac{18\cdot(3-1)^3}{12\cdot 10^2} = \dfrac{3}{25} = 0.12$

(We are within 0.12 of the true value!)

Example. If we use the midpoint approximation with $n = 20$ to estimate $\int_0^1 \sin(2x)\,dx$, how accurate are we guaranteed to be?

- $f(x) = \sin(2x)$, $f'(x) = 2\cos(2x)$, $f''(x) = -4\sin(2x)$
- On $[0,1]$, $|f''(x)| \le |-4\sin(2x)| \le 4$ since $|\sin(2x)| \le 1$
- So, $|E_M| \le \dfrac{4\cdot(1-0)^3}{24\cdot 20^2} = \dfrac{1}{2400}$

(We are within 0.00042 of the true value!)

Example. How large should n be to guarantee that using T_n to estimate $\int_0^1 e^{-3x}\,dx$ gives an error no larger than .001?

- $f(x) = e^{-3x}$, $f'(x) = -3e^{-3x}$, $f''(x) = 9e^{-3x}$
- On $[0,1]$, $f''(x)$ is largest at the left endpoint since $9e^{-3x}$ is decreasing $\Rightarrow |f''(x)| \le 9$
- Solve $|E_T| \le \dfrac{9\cdot 1^3}{12\cdot n^2} < 0.001 \Rightarrow n^2 > \dfrac{9}{12\cdot 0.001} = 750 \Rightarrow n > \sqrt{750} = 27.4$

$\boxed{n = 28}$

Web Assign

$$|E_M| \le \frac{k(b-a)^3}{24n^2}$$

Example. Consider the following table of values.

midpoints

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	2	3	4	6	7	9	10	10	11

(a) Use the midpoint rule with $n = 4$ to estimate $\int_0^4 f(x)\,dx$.

(b) Suppose that $-2 \le f''(x) \le 3$ for all x in $[0,4]$. Use the midpoint error bound to estimate the error.

(a) $$\Delta x = \frac{b-a}{n} = \frac{4-0}{4} = 1$$

$$M_4 = 1 \cdot (3+6+9+10) = 28$$

(b) $$|E_M| \le \frac{k\cdot(b-a)^3}{24n^2} = \frac{3\cdot(4-0)^3}{24\cdot 4^2} = \frac{3\cdot 4}{24} = \frac{1}{2}$$

(We can guarantee $27.5 \le \int_0^4 f(x)\,dx \le 28.5$)