

7.4 Integration of Rational Functions by Partial Fractions

Question. What is the goal of a partial fraction decomposition?

Question. What is a proper rational function? What is an improper rational function?

Question. Partial fractions only apply to proper fractions. What if $\frac{P(x)}{Q(x)}$ is improper?

Theorem (Partial Fraction Decomposition). Any proper rational function $\frac{P(x)}{Q(x)}$ can be rewritten as a sum of simpler fractions, called partial fractions. The decomposition is built from the following components based on the factors of $Q(x)$:

1. **Distinct Linear Factors:** For each distinct linear term $(a_1x + b_1)$ in $Q(x)$, the partial fraction decomposition includes a term of the form:

$$\frac{A_1}{a_1x + b_1}.$$

2. **Repeated Linear Factors:** For each repeated linear factor $(a_1x + b_1)^m$ in $Q(x)$, the partial fraction decomposition includes terms of the form:

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_m}{(a_1x + b_1)^m}.$$

3. **Irreducible Quadratic Factors:** For each irreducible quadratic term $(a_1x^2 + b_1x + c_1)$ in $Q(x)$, where $b_1^2 - 4a_1c_1 < 0$, the partial fraction decomposition includes a term of the form:

$$\frac{A_1x + B_1}{a_1x^2 + b_1x + c_1}.$$

4. **Repeated Irreducible Quadratic Factors:** For each repeated irreducible quadratic factor $(a_1x^2 + b_1x + c_1)^m$ in $Q(x)$, the partial fraction decomposition includes terms of the form:

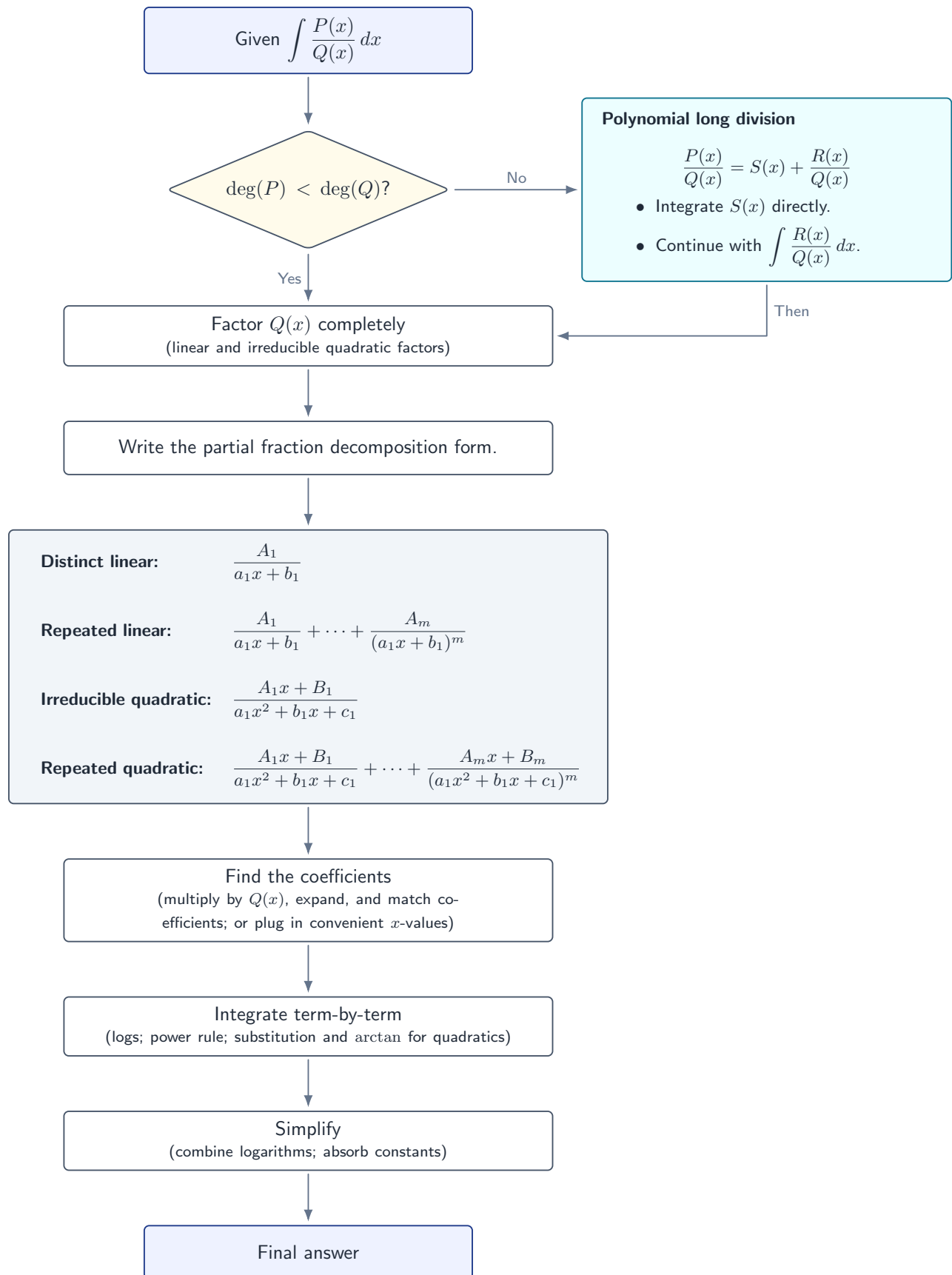
$$\frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{(a_1x^2 + b_1x + c_1)^2} + \cdots + \frac{A_mx + B_m}{(a_1x^2 + b_1x + c_1)^m}.$$

Example. Evaluate the partial fraction decomposition of $\frac{3x^2 + 5x + 2}{(x - 1)(x + 2)(x - 3)}$.

Example. Evaluate the partial fraction decomposition of $\frac{4x^2 + 7}{(x + 1)^2(x - 2)}$.

Example. Evaluate the partial fraction decomposition of $\frac{2x^2 + 3x + 1}{(x^2 + x + 1)(x - 1)}$.

Example. Evaluate the partial fraction decomposition of $\frac{5x^3 + 2x^2 + x + 4}{(x^2 + 1)^2(x - 2)}$.



Example. Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$.

Example. Find $\int \frac{dx}{x^2 - 9}$.

Example. Find $\int \frac{x^4 + x^3 + 6x^2 + 3x + 4}{x^3 + 4x} dx$.

Example. Write out the form of the partial fraction decomposition of the function

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2 + x + 1)(x^2 + 1)^3}$$

Example. Should we use partial fractions to solve $\int \frac{x^2 + 1}{x(x^2 + 3)} dx$?