e.g. $\frac{3x+5}{2x^2-5x-3} = \frac{2}{x-3} - \frac{1}{2x+1}$

7.4 Integration of Rational Functions by Partial Fractions

Question. What is the goal of a partial fraction decomposition?

Write $\frac{P(x)}{Q(x)}$ as a sum of simpler fractions (that we can integrate)

Question. What is a proper rational function? What is an improper rational function?

Proper: $\deg P(x) < \deg Q(x)$. Improper: $\deg P(x) \geq \deg Q(x)$

Question. Partial fractions only apply to proper fractions. What if $\frac{P(x)}{Q(x)}$ is improper?

Apply polynomial long division. $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$

where $S(x)$ is a polynomial and $R(x)/Q(x)$ is proper.

Theorem (Partial Fraction Decomposition)**.** Any proper rational function $\frac{P(x)}{Q(x)}$ can be rewritten as a sum of simpler fractions, called partial fractions. The decomposition is built from the following components based on the factors of $Q(x)$:

1. **Distinct Linear Factors:** For each distinct linear term (a_1x+b_1) in $Q(x)$, the partial fraction decomposition includes a term of the form: $(x-1)$

$$\frac{A_1}{a_1x+b_1}.$$

2. **Repeated Linear Factors:** For each repeated linear factor $(a_1x+b_1)^m$ in $Q(x)$, the partial fraction decomposition includes terms of the form: $(x-1)^3$

$$\frac{A_1}{a_1x+b_1} + \frac{A_2}{(a_1x+b_1)^2} + \cdots + \frac{A_m}{(a_1x+b_1)^m}.$$

3. **Irreducible Quadratic Factors:** For each irreducible quadratic term $(a_1x^2+b_1x+c_1)$ in $Q(x)$, where $b_1^2 - 4a_1c_1 < 0$, the partial fraction decomposition includes a term of the form: (x^2+1)

$$\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}.$$

4. **Repeated Irreducible Quadratic Factors:** For each repeated irreducible quadratic factor $(a_1x^2+b_1x+c_1)^m$ in $Q(x)$, the partial fraction decomposition includes terms of the form:

$(x^2+1)^4$

$$\frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \frac{A_2x+B_2}{(a_1x^2+b_1x+c_1)^2} + \cdots + \frac{A_mx+B_m}{(a_1x^2+b_1x+c_1)^m}.$$

Example. Evaluate the partial fraction decomposition of $\frac{3x^2+5x+2}{(x-1)(x+2)(x-3)}$.

$$= \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$$

Eventually we will solve for A, B, and C

Example. Evaluate the partial fraction decomposition of $\frac{4x^2+7}{(x+1)^2(x-2)}$.

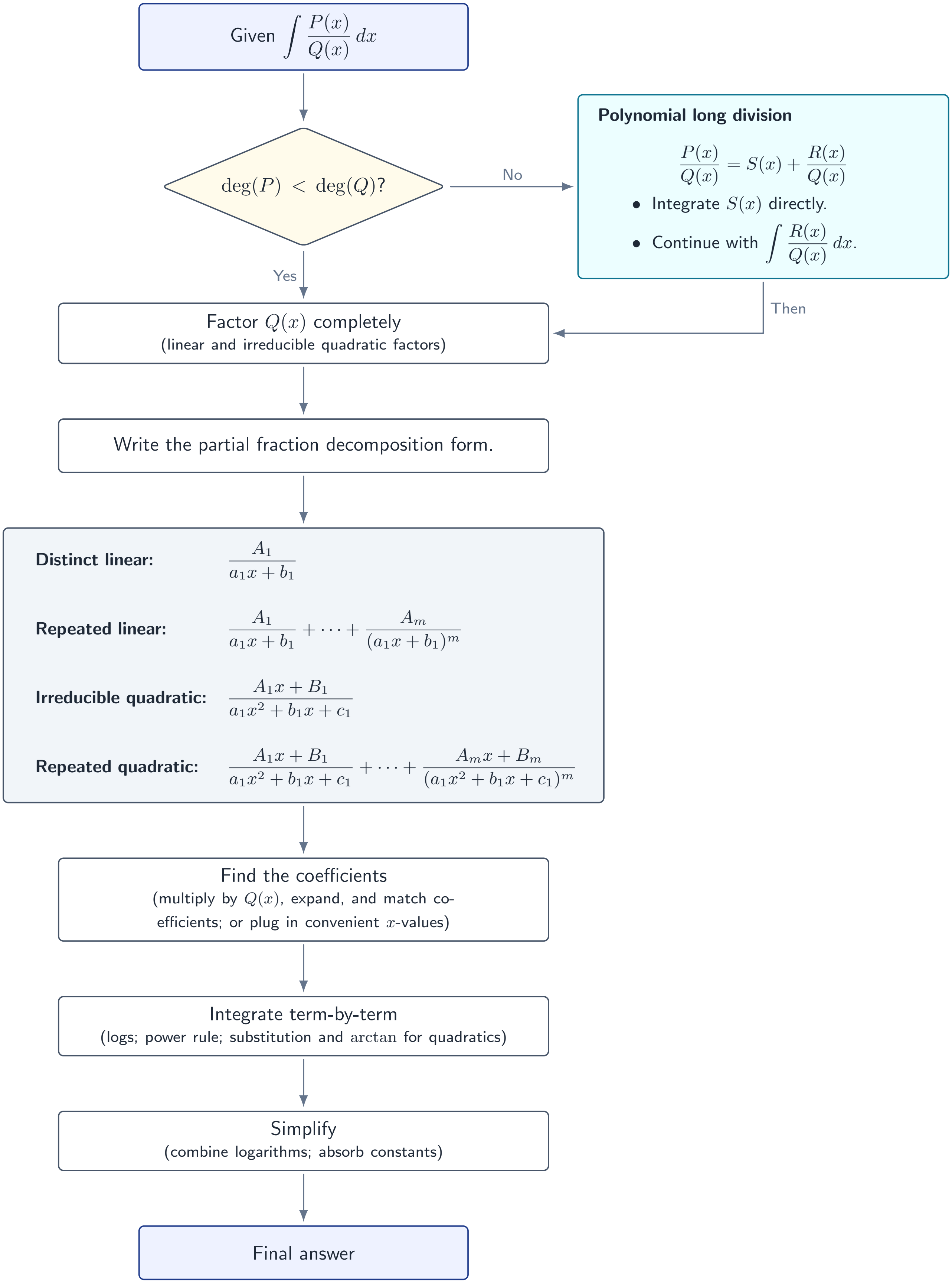
$$= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$$

Example. Evaluate the partial fraction decomposition of $\frac{2x^2+3x+1}{(x^2+x+1)(x-1)}$.

$$= \frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}$$

Example. Evaluate the partial fraction decomposition of $\frac{5x^3+2x^2+x+4}{(x^2+1)^2(x-2)}$.

$$= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-2}$$


Given $\int \frac{P(x)}{Q(x)}\,dx$
$\deg(P) < \deg(Q)$?
No
Polynomial long division
$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$
• Integrate $S(x)$ directly.
• Continue with $\int \frac{R(x)}{Q(x)}\,dx$.
Then
Yes
Factor $Q(x)$ completely
(linear and irreducible quadratic factors)
Write the partial fraction decomposition form.
Distinct linear: $\frac{A_1}{a_1x+b_1}$
Repeated linear: $\frac{A_1}{a_1x+b_1} + \cdots + \frac{A_m}{(a_1x+b_1)^m}$
Irreducible quadratic: $\frac{A_1x+B_1}{a_1x^2+b_1x+c_1}$
Repeated quadratic: $\frac{A_1x+B_1}{a_1x^2+b_1x+c_1} + \cdots + \frac{A_mx+B_m}{(a_1x^2+b_1x+c_1)^m}$
Find the coefficients
(multiply by $Q(x)$, expand, and match coefficients; or plug in convenient x-values)
Integrate term-by-term
(logs; power rule; substitution and $\arctan$ for quadratics)
Simplify
(combine logarithms; absorb constants)
Final answer

Example. Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.

① This is proper ✓

② Factor the denominator:

$$2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2)$$

③ Partial Fraction Decomposition

$$x\cdot(2x-1)(x+2)\cdot\frac{x^2+2x-1}{x(2x-1)(x+2)} = \left(\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\right)\cdot x(2x-1)(x+2)$$

④ Determine the coefficients (Multiply both sides by $Q(x)$)

$$\begin{aligned} x^2+2x-1 &= A(2x-1)(x+2) + B(x)(x+2) + C(x)(2x-1) \\ &= A(2x^2+3x-2) + B(x^2+2x) + C(2x^2-x) \\ &= (2A+B+2C)x^2 + (3A+2B-C)x - 2A \end{aligned}$$

Equate coefficients:

$$\begin{aligned} 2A+B+2C &= 1 \quad &\text{(coefficients of } x^2\text{)} \\ 3A+2B-C &= 2 \quad &\text{(coefficients of } x\text{)} \\ -2A &= -1 \quad &\text{(constant terms)} \end{aligned}$$

Solving the system, $A = 1/2$, $B = 1/5$, $C = -1/10$

⑤ $$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \frac{1/2}{x} + \frac{1/5}{2x-1} - \frac{1/10}{x+2}\,dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C$$

Example. Find $\int \frac{dx}{x^2-9}$.

① This is proper ✓

② Factor the denominator: $x^2-9 = (x-3)(x+3)$

③ Partial Fraction Decomposition:

$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$

④ Find coefficients (multiply through by $Q(x)$)

$$1 = A(x+3) + B(x-3)$$

Setting $x=3$ gives $A = 1/6$. Setting $x=-3$ gives $B = -1/6$.

⑤ $$\int \frac{1}{x^2-9}\,dx = \int \frac{1/6}{x-3} - \frac{1/6}{x+3}\,dx$$

$$= \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + C$$

$$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$$

Example. Find $\int \frac{x^4 + x^3 + 6x^2 + 3x + 4}{x^3 + 4x}\, dx$.

① This is improper since deg $P(x)$ > deg $Q(x)$. Polynomial long division:

$$\begin{array}{r|l} & x + 1 \\ \hline x^3+4x & x^4 + x^3 + 6x^2 + 3x + 4 \\ - & x^4 + 0x^3 + 4x^2 \\ \hline & x^3 + 2x^2 + 3x + 4 \\ - & x^3 + 0x^2 + 4x \\ \hline & 2x^2 - x + 4 \end{array}$$

$$\Rightarrow \frac{P(x)}{Q(x)} = \underbrace{x+1}_{\text{easy to integrate}} + \underbrace{\frac{2x^2 - x + 4}{x^3 + 4x}}_{\text{proper}}$$

② Factor the denominator: $x^3 + 4x = x(x^2+4)$

irreducible $b^2 - 4ac < 0$

③ Partial Fraction Decomposition

$$x \cdot (x^2+4) \cdot \frac{2x^2 - x + 4}{x(x^2+4)} = \left(\frac{A}{x} + \frac{Bx + C}{x^2+4} \right) \cdot x(x^2+4)$$

(4) Find the coefficients (multiply through by $Q(x)$)

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$$

$$= Ax^2 + 4A + Bx^2 + Cx$$

$$= (A+B)x^2 + Cx + 4A$$

Equate coefficients:

$$A + B = 2$$
$$C = -1$$
$$4A = 4$$

$$\Rightarrow \quad A = 1, \; B = 1, \; C = -1$$

(5) $$\int \frac{x^4 + x^3 + 6x^2 + 3x + 4}{x^3 + 4x}\,dx = \int x + 1 + \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$$

$$= \int x + 1 + \frac{1}{x} + \frac{x-1}{x^2+4}\,dx$$

$$= \int x + 1 + \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{2}x^2 + x + \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Exam style (M.C.)

Example. Write out the form of the partial fraction decomposition of the function

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

Example. Should we use partial fractions to solve $\int \frac{x^2+1}{x(x^2+3)}\,dx$?

We could, u-substitution is simpler.

Let $u = x(x^2+3) = x^3+3x$

Then $du = 3x^2+3 = 3(x^2+1)$

$$\Rightarrow \int \frac{1}{u}\cdot\frac{1}{3}\,du = \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|x^3+3x| + C$$