7.2 Trigonometric Integrals

Integrals of Powers of Sine and Cosine

We begin by considering integrals in which the integrand is a power of sine, a power of cosine, or a product of these.

$$\int \sin^m x \cos^n x \, dx$$

m odd

Save a factor of $\sin x$ to create du:

$$\sin^m x \cos^n x = \sin^{m-1} x \cos^n x \, (\sin x).$$

Rewrite $\sin^{m-1} x$ using $\sin^2 x = 1 - \cos^2 x$, then use

$$u = \cos x, \qquad du = -\sin x \, dx.$$

n odd

Save a factor of $\cos x$ to create du:

$$\sin^m x \cos^n x = \sin^m x \cos^{n-1} x \, (\cos x).$$

Rewrite $\cos^{n-1} x$ using $\cos^2 x = 1 - \sin^2 x$, then use

$$u = \sin x, \qquad du = \cos x \, dx.$$

m, n even

If m and n are both even, reduce powers using

$$\sin^2 x = \frac{1 - \cos(2x)}{2}, \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}.$$

Then simplify and integrate term-by-term. A useful identity is

$$\sin x \cos x = \frac{1}{2} \sin(2x).$$

Idea: Try to *create the derivative you want* and then use identities to rewrite whatever is left.

- If you can save a factor of $\cos x$, then $u = \sin x$ and you rewrite any remaining even powers of $\cos x$ using $\cos^2 x = 1 - \sin^2 x$.
- If you can save a factor of $\sin x$, then $u = \cos x$ and you rewrite any remaining even powers of $\sin x$ using $\sin^2 x = 1 - \cos^2 x$.

* Need to know these trig identities.

Example. Find $\int \sin^5 x \cos^2 x \, dx$.

$$u = \sin x \to \sin^5 x \cos x \cdot \underbrace{\cos x \, dx}_{du}$$

$$u = \cos x \to \sin^4 x \cos^2 x \cdot \underbrace{\sin x \, dx}_{du} \quad \checkmark$$

Easily converts

$$\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \cdot \cos^2 x \cdot \sin x \, dx$$

$$= \int (\sin^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx$$

$$= \int (1 - \cos^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx$$

Let $u = \cos x$. Then $du = -\sin x \, dx$.

$$-\int (1-u^2)^2 \cdot u^2 \, du = -\int (1 - 2u^2 + u^4) \cdot u^2 \, du$$

$$= -\int u^2 - 2u^4 + u^6 \, du$$

$$= -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right) + C$$

$$= -\left(\frac{\cos^3 x}{3} - \frac{2\cos^5 x}{5} + \frac{\cos^7 x}{7} \right) + C$$

Example. Find $\int \sin^4 x\,dx$. All even powers $\rightarrow$ reduce using half-angle

$$\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx$$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx$$

$$= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x)\,dx$$

Half angle identity: $\cos^2(2x) = \dfrac{1+\cos(4x)}{2} = \frac{1}{2}(1+\cos(4x))$

$$= \frac{1}{4}\int 1 - 2\cos(2x) + \frac{1}{2}(1+\cos(4x))\,dx$$

$$= \frac{1}{4}\int \frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\,dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x)\right) + C$$

Integrals of Powers of Secant and Tangent

Now we consider integrals in which the integrand is a power of tangent, a power of secant, or a product of these.

$$\int \tan^m x \sec^n x \, dx$$

n even →

Save a factor of $\sec^2 x$ to create du:

$$\tan^m x \sec^n x = \tan^m x \sec^{n-2} x \sec^2 x.$$

Use $\sec^2 x = 1 + \tan^2 x$ to rewrite the remaining $\sec^{n-2} x$, then substitute.

m odd →

Save a factor of $\sec x \tan x$ to create du:

$$\tan^m x \sec^n x = \tan^{m-1} x \sec^{n-1} x \,(\sec x \tan x).$$

Use $\tan^2 x = \sec^2 x - 1$ to rewrite the remaining $\tan^{m-1} x$, then substitute.

m even, n odd →

Harder case with no single standard trick. General ideas: rewrite in terms of $\sin x$ and $\cos x$

$$\tan x = \frac{\sin x}{\cos x}, \qquad \sec x = \frac{1}{\cos x},$$

then try $u = \sin x$ or $u = \cos x$. Alternatively, convert everything to $\sec x$ and use integration by parts.

Need to know: $1 + \tan^2 x = \sec^2 x$

If you forget: $\cos^2 x + \sin^2 x = 1$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$1 + \tan^2 x = \sec^2 x$$

Idea: Try to *create the derivative you want* and then use identities to rewrite whatever is left.

- If you can save a factor of $\sec x \tan x$, then $u = \sec x$ and you rewrite any remaining even powers of $\tan x$ using $\tan^2 x = \sec^2 x - 1$.
- If you can save a factor of $\sec^2 x$, then $u = \tan x$ and you rewrite any remaining even powers of $\sec x$ using $\sec^2 x = 1 + \tan^2 x$.

Example. Evaluate $\int \tan^6 x \sec^4 x\, dx$.

$u = \sec x$
$du = \sec x \tan x\, dx$

$\tan^5 x \sec^3 x \cdot \underbrace{\sec x \tan x}_{du}$

$u = \tan x$
$du = \sec^2 x\, dx$

$\tan^6 x \sec^2 x \cdot \underbrace{\sec^2 x}_{du}$ ✓

easily converts to tan

$$\int \tan^6 x \sec^4 x\, dx = \int \tan^6 x \cdot \sec^2 x \cdot \sec^2 x\, dx$$

$$= \int \tan^6 x \cdot (1 + \tan^2 x) \cdot \sec^2 x\, dx$$

Let $u = \tan x$. Then $du = \sec^2 x\, dx$

$$= \int u^6 \cdot (1 + u^2)\, du$$

$$= \int u^6 + u^8\, du$$

$$= \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^9 x}{9} + C$$

Example. Find $\int \tan^5\theta \sec^7\theta \, d\theta$.

easily converts

$u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$ $\longrightarrow$ $\tan^4\theta\, \sec^6\theta\, \overbrace{\sec\theta\, \tan\theta}^{du}$ ✓✓

$u = \tan\theta$, $du = \sec^2\theta\, d\theta$ $\longrightarrow$ $\tan^5\theta\, \sec^5\theta\, \overbrace{\sec^2\theta}^{du}$

$$\int \tan^5\theta \sec^7\theta\, d\theta = \int \tan^4\theta \sec^6\theta \cdot \sec\theta\tan\theta\, d\theta$$

$$= \int (\tan^2\theta)^2 \sec^6\theta \cdot \sec\theta \tan\theta\, d\theta$$

$$= \int (\sec^2\theta - 1)^2 \sec^6\theta \cdot \sec\theta\tan\theta\, d\theta$$

Let $u = \sec\theta$. Then $du = \sec\theta\tan\theta\, d\theta$

$$= \int (u^2-1)^2 \cdot u^6\, du$$

$$= \int (u^4 - 2u^2 + 1)\cdot u^6\, du$$

$$= \int u^{10} - 2u^8 + u^6\, du$$

$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C$$

$$= \frac{\sec^{11}\theta}{11} - \frac{2\sec^9\theta}{9} + \frac{\sec^7\theta}{7} + C$$

Remark. In some cases, the guidelines for integrating powers of $\tan x$ and $\sec x$ are not as straightforward. We may need to use trigonometric identities, integration by parts, or creative problem-solving techniques. Two important integrals to remember for these cases are:

$$\int \tan x \, dx = \ln|\sec x| + C$$

and

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C.$$

Example. Find $\int \tan^3 x \, dx$.

We can't pull out a $\sec^2 x$ or a $\sec x \tan x$ term.

Rewrite as $\tan^2 x \cdot \tan x$ and use $\tan^2 x = \sec^2 x - 1$

$$\begin{aligned}
\int \tan^3 x \, dx &= \int \tan x \cdot \tan^2 x \, dx \\
&= \int \tan x \cdot (\sec^2 x - 1) \, dx \\
&= \int \tan x \cdot \sec^2 x - \tan x \, dx \\
&= \int \tan x \sec^2 x \, dx - \int \tan x \, dx \\
&= \frac{1}{2} \tan^2 x - \ln|\sec x| + C
\end{aligned}$$

$u = \tan x$

$du = \sec^2 x$

$\int u \, du = \frac{1}{2} u^2$

Boomerang!

Example. Find $\int \sec^3 x\,dx$.

$u = \sec x$ $\quad$ $v = \tan x$

$du = \sec x \tan x\,dx$ $\quad$ $dv = \sec^2 x\,dx$

$$\underbrace{\int \sec^3 x\,dx}_{u \cdot dv} = \underbrace{\sec x \tan x}_{u \cdot v} - \int \underbrace{\tan^2 x \sec x\,dx}_{v \cdot du}$$

$$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx$$

$$= \sec x \tan x - \int \sec^3 x - \sec x\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|$$

Solve for $\int \sec^3 x\,dx$:

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$