7.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called *integration by parts*.

Theorem (Integration by Parts, Formula 1)**.** If f and g are differentiable functions, then

$$\int f(x)\,g'(x)\,dx \;=\; f(x)\,g(x) \;-\; \int f'(x)\,g(x)\,dx.$$

Proof.

1. Recall the Product Rule.

2. Integrate both sides.

3. Rearrange to isolate $\int f(x)g'(x)\,dx$.

We usually omit "$+C$" in the integration by parts formula, since any constants can be absorbed into one; when applying the formula, add a single "$+C$" to the final antiderivative.

□

The formula for integration by parts is often written in the following form:

Theorem (Integration by Parts, Formula 2)**.** Let $u = f(x)$ and $v = g(x)$ in the theorem above. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. We obtain:

$$\int u\,dv \;=\; u\,v \;-\; \int v\,du.$$

Choosing u and v. Integration by parts is most useful when differentiating one factor simplifies it, while integrating the other factor is still doable. A common guideline is **LIATE**, which ranks function types from "best choice for u" to "worst choice for u":

$$\mathbf{L}\ (\ln,\ \text{logs}) > \mathbf{I}\ (\text{inverse trig}) > \mathbf{A}\ (\text{algebraic / polynomials}) > \mathbf{T}\ (\text{trig}) > \mathbf{E}\ (\text{exponential}).$$

In a product $\int f(x)g(x)\,dx$, pick u to be the factor that appears *earliest* in LIATE (so u gets simpler when differentiated). The remaining factor becomes dv, so that $v = \int dv$ is something you can actually compute.

Example. Find $\displaystyle\int x \sin x\, dx$.

Question. In the above example, what would have happened if we had instead chosen $u = \sin x$ and $dv = x\,dx$?

Example. Evaluate $\int x^2 \ln x \, dx$.

Example. Evaluate $\int \ln x \, dx$.

Example. Find $\int t^2 e^t \, dt$.

Example. Evaluate $\int e^x \sin x \, dx$.

Definite Integration by Parts

If we combine the formula for integration by parts with the Evaluation Theorem, we can evaluate definite integrals by parts:

Theorem (Definite Integration by Parts)**.**

$$\int_a^b f(x)\, g'(x)\, dx \;=\; \Big[f(x)\, g(x)\Big]_a^b \;-\; \int_a^b f'(x)\, g(x)\, dx.$$

Example. Calculate $\displaystyle\int_0^1 \tan^{-1}(x)\, dx$.