## 7.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called *integration by parts*.

**Theorem** (Integration by Parts, Formula 1)**.** If $f$ and $g$ are differentiable functions, then

$$\int f(x)\,g'(x)\,dx \;=\; f(x)\,g(x) \;-\; \int f'(x)\,g(x)\,dx.$$

*Proof.*

1. Recall the Product Rule.

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

2. Integrate both sides.

$$\int \frac{d}{dx}\left[f(x)\cdot g(x)\right] dx = \int f'(x)g(x) + f(x)g'(x)\,dx$$

$$\Rightarrow \quad f(x)\cdot g(x) + C = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

3. Rearrange to isolate $\int f(x)g'(x)\,dx$.

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx + C$$

We usually omit "$+C$" in the integration by parts formula, since any constants can be absorbed into one; when applying the formula, add a single "$+C$" to the final antiderivative.

□

The formula for integration by parts is often written in the following form:

**Theorem** (Integration by Parts, Formula 2)**.** Let $u = f(x)$ and $v = g(x)$ in the theorem above. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. We obtain:

$$\int u\,dv \;=\; u\,v \;-\; \int v\,du.$$

In Practice: ① choose u
② Choose dv
③ Compute the R.H.S.

Ultra violet voodoo?

"Choose $u$ so that you differentiate what gets simpler"

**Choosing $u$ and $v$.** Integration by parts is most useful when differentiating one factor simplifies it, while integrating the other factor is still doable. A common guideline is **LIATE**, which ranks function types from "best choice for $u$" to "worst choice for $u$":

**L** (ln, logs) > **I** (inverse trig) > **A** (algebraic / polynomials) > **T** (trig) > **E** (exponential).

$x^2$, $2x$ $\qquad$ $e^x$

In a product $\int f(x)g(x)\,dx$, pick $u$ to be the factor that appears *earliest* in LIATE (so $u$ gets simpler when differentiated). The remaining factor becomes $dv$, so that $v = \int dv$ is something you can actually compute.

LIATE

**Example.** Find $\int x \sin x\, dx$.

$$\int u\,dv = u\cdot v - \int v\,du$$

$u = x \qquad v = -\cos x$

$du = dx \qquad dv = \sin x \cdot dx$

$$\int \overbrace{x\cdot \sin x\, dx}^{u\cdot dv} = \overbrace{x\cdot(-\cos x)}^{u\cdot v} - \int \overbrace{-\cos x \cdot dx}^{v\cdot du}$$

$$= -x\cos x + \int \cos x\, dx$$

$$= -x\cos x + \sin x + C$$

**Question.** In the above example, what would have happened if we had instead chosen $u = \sin x$ and $dv = x\,dx$?

Here, $du = \cos x\, dx$ and $v = \int x\,dx = \frac{x^2}{2}$. Then

$$\int \overbrace{x \sin x\, dx}^{u\cdot dv} = \overbrace{\sin x \cdot \frac{x^2}{2}}^{u\cdot v} - \int \overbrace{\frac{x^2}{2}\cdot \cos x\, dx}^{v\cdot du}$$

This is more complicated than the original integral!

LIATE

**Example.** Evaluate $\int x^2 \ln x \, dx$.

$\int u\,dv = u \cdot v - \int v\,du$

$u = \ln(x)$ $\qquad v = \frac{1}{3}x^3$

$du = \frac{1}{x}dx$ $\qquad dv = x^2 dx$

$$\underbrace{\int x^2 \cdot \ln(x)\,dx}_{u \cdot dv} = \underbrace{\ln(x) \cdot \frac{1}{3}x^3}_{u \cdot v} - \int \underbrace{\frac{1}{3}x^3 \cdot \frac{1}{x}dx}_{v \cdot du}$$

$$= \ln(x) \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^2\,dx$$

$$= \ln(x) \cdot \frac{1}{3}x^3 - \frac{1}{9}x^3 + C$$

**Example.** Evaluate $\int \ln x \, dx$.

* Sometimes we can set $dv = dx$

(typically for log / inverse trig)

$u = \ln(x)$ $\qquad v = x$

$du = \frac{1}{x}dx$ $\qquad dv = dx$

$$\underbrace{\int \ln(x)dx}_{u \cdot dv} = \underbrace{\ln(x) \cdot x}_{u \cdot v} - \int \underbrace{x \cdot \frac{1}{x}dx}_{v \cdot du}$$

$$= x\ln(x) - \int 1\,dx$$

$$= x\ln(x) - x + C$$

$$\overset{\text{LIATE}}{}$$

**Example.** Find $\int t^2 e^t \, dt$.

Resilience

Integration by Parts:

$$\overset{u\cdot dv}{\int t^2 \cdot e^t \, dt} = \overset{u\cdot v}{t^2 e^t} - \overset{v\cdot du}{\int e^t \cdot 2t\, dt}$$

$u = t^2$ $\quad$ $v = e^t$

$du = 2t\,dt$ $\quad$ $dv = e^t\,dt$

Integration by Parts (again):

$$\overset{u\cdot dv}{\int e^t \cdot 2t\, dt} = \overset{u\cdot v}{2t \cdot e^t} - \overset{v\cdot du}{\int e^t \cdot 2\, dt}$$

$$= 2t\cdot e^t - 2e^t + C$$

$u = 2t$ $\quad$ $v = e^t$

$du = 2\,dt$ $\quad$ $dv = e^t\,dt$

In total,

$$\int t^2 \cdot e^t \, dt = t^2 \cdot e^t - \left(2te^t - 2e^t + C\right)$$

$$= t^2 e^t - 2te^t + 2e^t + C$$

$$= e^t (t^2 - 2t + 2) + C$$

"Boomerang Integral"

Common free response question on exams

**Example.** Evaluate $\int e^x \sin x \, dx$.

LIATE

$u = \sin x$ $\quad v = e^x$

$du = \cos x \cdot dx$ $\quad dv = e^x dx$

$$\underbrace{\int e^x \cdot \sin x \, dx}_{u \cdot dv} = \underbrace{e^x \cdot \sin x}_{u \cdot v} - \underbrace{\int e^x \cdot \cos x \, dx}_{v \cdot du}$$

$u = \cos x$ $\quad v = e^x$

$du = -\sin x \, dx$ $\quad dv = e^x \, dx$

$$\int e^x \cos x \, dx = e^x \cdot \cos x - \int e^x \cdot (-\sin x) \, dx$$

$$= e^x \cdot \cos x + \int e^x \sin x \, dx$$

In total,

$$\int e^x \cdot \sin x \, dx = e^x \cdot \sin x - \left( e^x \cos x + \int e^x \sin x \, dx \right)$$

$$= e^x \cdot \sin x - e^x \cos x - \int e^x \sin x \, dx$$

So

$$2 \cdot \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\Rightarrow \int e^x \sin x \, dx = \frac{e^x \sin x - e^x \cos x}{2} + C$$

* Remember to add the $+ C$

**Definite Integration by Parts**

If we combine the formula for integration by parts with the Evaluation Theorem, we can evaluate definite integrals by parts:

**Theorem** (Definite Integration by Parts)**.**

$$\int_a^b \overset{u \cdot dv}{f(x)\, g'(x)}\, dx \;=\; \overset{u \cdot v}{\Big[f(x)\, g(x)\Big]_a^b} \;-\; \int_a^b \overset{v \cdot du}{f'(x)\, g(x)}\, dx.$$

**Example.** Calculate $\displaystyle\int_0^1 \tan^{-1}(x)\, dx$.

$u = \tan^{-1}(x)$ $\quad v = x$

$du = \dfrac{1}{1+x^2}\, dx$ $\quad dv = dx$

① $\displaystyle\int_0^1 \overset{u \cdot dv}{\tan^{-1}(x)\, dx} = \overset{u \cdot v}{\Big[\tan^{-1}(x) \cdot x\Big]_0^1} - \int_0^1 \overset{v \cdot du}{\frac{x}{1+x^2}\, dx}$

② $\Big[\tan^{-1}(x) \cdot x\Big]_0^1 = \tan^{-1}(1) \cdot 1 - \tan^{-1}(0) \cdot 0 = \dfrac{\pi}{4}$

③ Compute $\displaystyle\int_0^1 \frac{x}{1+x^2}\, dx$. Let $u = 1+x^2$. Then $du = 2x \cdot dx$ $\Rightarrow \frac{1}{2} du = x\, dx$

when $x=0$, $u=1$

when $x=1$, $u=2$

$$\int_{x=0}^{x=1} \frac{x}{1+x^2}\, dx = \int_{u=1}^{u=2} \frac{1}{u} \cdot \frac{1}{2}\, du = \Big[\tfrac{1}{2} \ln(|u|)\Big]_{u=1}^{u=2} = \frac{1}{2}\ln(2)$$

④ In total, $\displaystyle\int_0^1 \tan^{-1}(x)\, dx = \frac{\pi}{4} - \frac{1}{2}\ln(2)$