7.1 Integration by Parts

Every differentiation rule has a corresponding integration rule. For instance, the Substitution Rule for integration corresponds to the Chain Rule for differentiation. The rule that corresponds to the Product Rule for differentiation is called *integration by parts*.

Theorem (Integration by Parts, Formula 1)**.** If f and g are differentiable functions, then

$$\int f(x)\,g'(x)\,dx \;=\; f(x)\,g(x) \;-\; \int f'(x)\,g(x)\,dx.$$

Proof.

1. Recall the Product Rule.

$$\frac{d}{dx}\Big[f(x)g(x)\Big] = f'(x)g(x) + f(x)g'(x)$$

2. Integrate both sides.

$$\int \frac{d}{dx}\big[f(x)g(x)\big]\,dx = \int f'(x)g(x) + f(x)g'(x)\,dx$$

$$\Rightarrow \quad f(x)g(x) + C = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

3. Rearrange to isolate $\int f(x)g'(x)\,dx$.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx + C$$

We usually omit "$+C$" in the integration by parts formula, since any constants can be absorbed into one; when applying the formula, add a single "$+C$" to the final antiderivative.

□

The formula for integration by parts is often written in the following form:

Theorem (Integration by Parts, Formula 2)**.** Let $u = f(x)$ and $v = g(x)$ in the theorem above. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. We obtain:

$$\int u\,dv \;=\; \underline{u}\,\underline{v} \;-\; \int \underline{v}\,\underline{du}.$$

In practice: ① Choose u
② Choose dv
③ Compute the R.H.S.

ultraviolet voodoo

" Choose u so that you differentiate what gets simpler "

Choosing u and v. Integration by parts is most useful when differentiating one factor simplifies it, while integrating the other factor is still doable. A common guideline is **LIATE**, which ranks function types from "best choice for u" to "worst choice for u":

L (ln, logs) > **I** (inverse trig) > **A** (algebraic / polynomials) > **T** (trig) > **E** (exponential).

$x^2, 2x$ e^x

In a product $\int f(x)g(x)\,dx$, pick u to be the factor that appears *earliest* in LIATE (so u gets simpler when differentiated). The remaining factor becomes dv, so that $v = \int dv$ is something you can actually compute.

LIATE

Example. Find $\int x \sin x \, dx$.

$$\int u\,dv = u \cdot v - \int v\,du$$

$u = x \qquad v = -\cos x$

$du = dx \qquad dv = \sin x \, dx$

$$\int \underbrace{x \sin x \, dx}_{u \cdot dv} = \underbrace{x \cdot (-\cos x)}_{u \cdot v} - \int \underbrace{(-\cos x)\,dx}_{v \cdot du}$$

$$= -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + C$$

Question. In the above example, what would have happened if we had instead chosen $u = \sin x$ and $dv = x\,dx$?

Here, $du = \cos(x)\,dx$ and $v = \int x\,dx = \frac{x^2}{2}$. Then

$$\int \underbrace{x \sin x \, dx}_{u \cdot dv} = \underbrace{\sin(x) \cdot \frac{x^2}{2}}_{u \cdot v} - \int \underbrace{\frac{x^2}{2} \cos(x)\,dx}_{v \cdot du}$$

This is more complicated than the original integral!

LIATE

Example. Evaluate $\int x^2 \ln x \, dx$.

$$\int u\,dv = u\cdot v - \int v\,du$$

$u = \ln(x)$ $\qquad$ $v = \frac{1}{3}x^3$

$du = \frac{1}{x}\,dx$ $\qquad$ $dv = x^2\,dx$

$$\overset{u\cdot dv}{\int x^2 \ln(x)\,dx} = \overset{u\cdot v}{\ln(x)\cdot \tfrac{1}{3}x^3} - \overset{v\cdot du}{\int \tfrac{1}{3}x^3 \cdot \tfrac{1}{x}\,dx}$$

$$= \ln(x)\cdot\tfrac{1}{3}x^3 - \int \tfrac{1}{3}x^2\,dx$$

$$= \ln(x)\cdot\tfrac{1}{3}x^3 - \tfrac{1}{9}x^3 + C$$

Example. Evaluate $\int \ln x \, dx$.

* sometimes we can set $dv = dx$

(typically for log / inverse trig)

$$\overset{u\cdot dv}{\int \ln(x)\,dx} = \overset{u\cdot v}{\ln(x)\cdot x} - \overset{v\cdot du}{\int x\cdot \tfrac{1}{x}\,dx}$$

$$= x\ln(x) - \int 1\,dx$$

$$= x\ln(x) - x + C$$

$u = \ln(x)$ $\qquad$ $v = x$

$du = \frac{1}{x}\,dx$ $\qquad$ $dv = dx$

LIATE

Resilience

Example. Find $\int t^2 e^t \, dt$.

Integration by Parts:

$$\int \underbrace{t^2}_{u} \cdot \underbrace{e^t \, dt}_{dv} = \underbrace{t^2 \cdot e^t}_{u \cdot v} - \int \underbrace{e^t \cdot 2t \, dt}_{v \cdot du}$$

$u = t^2$ $\quad$ $v = e^t$

$du = 2t \, dt$ $\quad$ $dv = e^t \, dt$

Integration by Parts (again):

$$\int \underbrace{e^t \cdot 2t \, dt}_{u \cdot dv} = 2t \cdot e^t - \int e^t \cdot 2 \, dt$$

$$= 2te^t - 2e^t + C$$

$u = 2t$ $\quad$ $v = e^t$

$du = 2 \, dt$ $\quad$ $dv = e^t \, dt$

In total,

$$\int t^2 \cdot e^t \, dt = t^2 \cdot e^t - \left(2te^t - 2e^t + C\right)$$

$$= t^2 e^t - 2te^t + 2e^t + C$$

$$= e^t \left(t^2 - 2t + 2\right) + C$$

Example. Evaluate $\int e^x \sin x \, dx$.

Definite Integration by Parts

If we combine the formula for integration by parts with the Evaluation Theorem, we can evaluate definite integrals by parts:

Theorem (Definite Integration by Parts)**.**

$$\int_a^b f(x)\, g'(x)\, dx \;=\; \Big[f(x)\, g(x)\Big]_a^b \;-\; \int_a^b f'(x)\, g(x)\, dx.$$

Example. Calculate $\displaystyle\int_0^1 \tan^{-1}(x)\, dx$.