6.5 Average Value of a Function

Theorem. The average value of f on the interval $[a, b]$ is $f_{\text{avg}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

- The average of n numbers $x_1, x_2, \ldots, x_n$ is given by:

$$\text{Average} = \frac{\text{Total Sum}}{\text{Number of values}} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

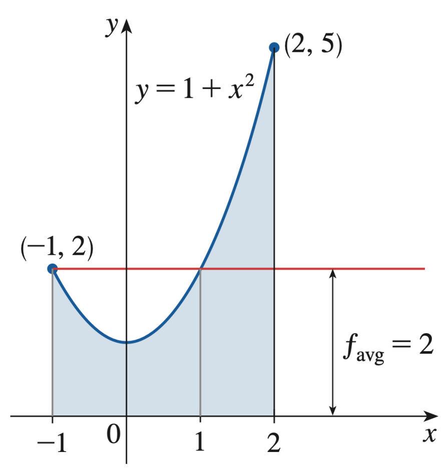
- For functions, the idea of "average" extends to infinitely many values (since a function takes on a value at every point in the interval $[a, b]$). We use an integral to represent the total sum.

$$\text{Total Sum} = \int_a^b f(x)\,dx$$

- The average value requires dividing the total sum by the "number of values." For functions over $[a, b]$, the equivalent is dividing by the length of the interval $b - a$.

$$f_{avg} = \frac{\text{Total Sum}}{\text{Number of values}} = \frac{\int_a^b f(x)\,dx}{b-a}$$

Example. Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.



$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2-(-1)}\int_{-1}^{2} 1+x^2\,dx$$

$$= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2}$$

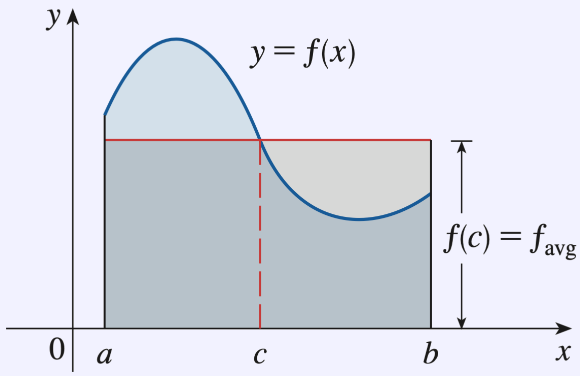
$$= \frac{1}{3}\left[\left(2+\frac{8}{3}\right) - \left(-1-\frac{1}{3}\right)\right]$$

$$= 2$$

Theorem (The Mean Value Theorem for Integrals)**.** If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

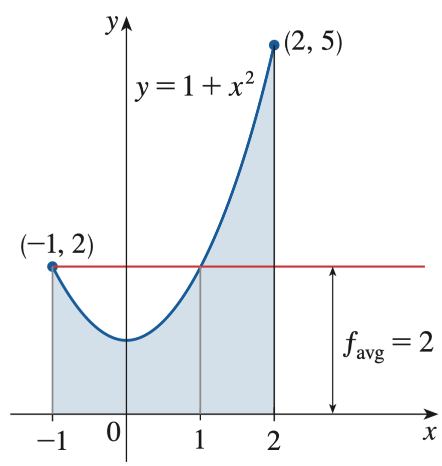
$$f(c) = f\text{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is, $\int_a^b f(x)\,dx = f(c)(b-a)$



You can always chop off the top of a mountain at a certain height (namely, f_{avg}), and use it to fill in the valleys so that the mountain is completely flat.

Example. Determine the value of c that satisfies the conclusion of the Mean Value Theorem for the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.



By the M.V.T., there is some c in $[-1, 2]$ so that

$$f(c) = f_{avg} = \frac{1}{2-(-1)}\int_{-1}^{2} 1+x^2\,dx = 2$$

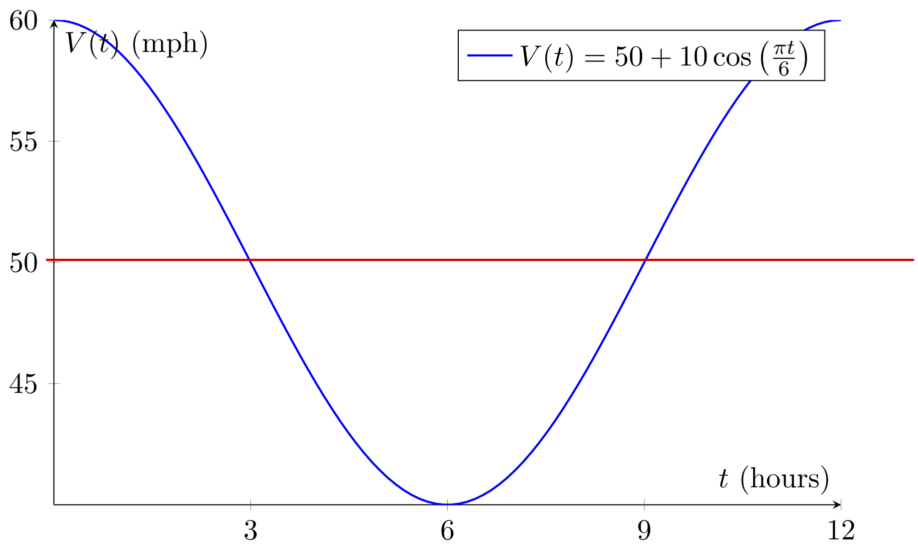
$\Rightarrow \quad 1 + c^2 = 2$

$\Rightarrow \quad c^2 = 1$

$\Rightarrow \quad c = \pm 1$

Example (Monitoring Traffic Flow)**.** Traffic engineers are analyzing vehicle speeds on a 6-mile stretch of highway during a 12-hour observation period. The speed of traffic (in miles per hour) at time t (in hours) is modeled as:

$$V(t) = 50 + 10\cos\left(\frac{\pi t}{6}\right), \quad t \in [0, 12].$$



$50\,\text{mph} = f_{avg}$

Question. Knowing the average speed helps ensure that traffic flows smoothly and within safe limits, preventing dangerous speed fluctuations. What is the average speed of vehicles over the 12-hour period?

$$V_{avg} = \frac{1}{b-a}\int_a^b V(t)\,dt = \frac{1}{12}\int_0^{12} 50 + 10\cos\left(\frac{\pi t}{6}\right) dt$$

$$= \frac{1}{12}\int_0^{12} 50\,dt + \frac{1}{12}\int_0^{12} 10\cos\left(\frac{\pi t}{6}\right) dt$$

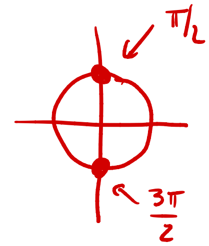
$$= \frac{1}{12}\Big[50t\Big]_0^{12} + \frac{1}{12}\left[\frac{60}{\pi}\sin\left(\frac{\pi t}{6}\right)\right]_0^{12}$$

$$= 50 \text{ mph}$$

Question. At what times does the instantaneous speed match the average speed? These moments indicate when traffic flow is most representative of overall conditions, helping engineers optimize traffic signals and safety measures.

By the mean value theorem, there is some c in $[0,12]$ so that $v(c) = V_{avg} = 50$

$$50 + 10\cos\left(\frac{\pi \cdot c}{6}\right) = 50$$



$$\Rightarrow \cos\left(\frac{\pi \cdot c}{6}\right) = 0$$

$$\Rightarrow \frac{\pi c}{6} = \frac{\pi}{2} + n\pi \quad \text{for all integers } n$$

$$\Rightarrow c = 3 + 6n \quad \text{for all integers } n$$

In $[0,12]$, $c = 3, 9$ $(n = 0, 1)$

- If this is afternoon traffic, the instantaneous speed matches the average speed at 3pm and 9pm
- This tells engineers when to observe traffic to get representative conditions.