

6.4 Work

Definition. In physics, a **force** is defined as a push or pull on an object. For example, a horizontal push of a book across a table, or the downward pull of gravity on a ball. Mathematically, if an object moves along a straight line with position function $s(t)$, the force F acting on the object is given by Newton's Second Law of Motion:

$$F = ma = m \frac{d^2 s}{dt^2},$$

where m is the mass of the object, and a is the acceleration of the object.

Definition. Work is defined as the product of the force F acting on an object and the distance d the object moves:

$$W = Fd \quad (\text{work} = \text{force} \times \text{distance}).$$

- If F is measured in newtons (N) and d in meters (m), W is measured in *joules* (J):

$$1 \text{ J} = 1 \text{ N} \cdot \text{m}.$$

- If F is measured in pounds (lb) and d in feet (ft), W is measured in *foot-pounds* (ft-lb):

$$1 \text{ ft-lb} \approx 1.36 \text{ J}.$$

Example.

- (a) How much work is done in lifting a 1.2-kg book off the floor to put it on a desk that is 0.7 m high? Use the fact that the acceleration due to gravity is $g = 9.8 \text{ m/s}^2$.
- (b) How much work is done in lifting a 20-lb weight 6 ft off the ground? ← Here, the force is already given (1b)

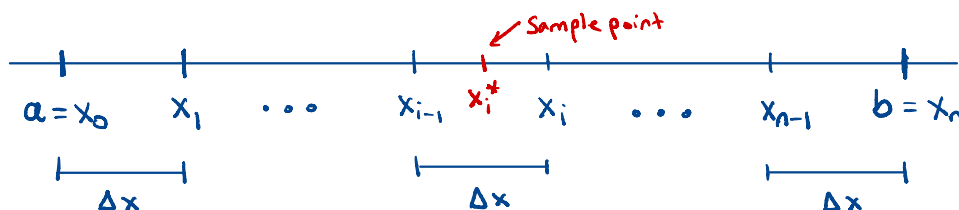
$$\begin{aligned} \text{(a)} \quad W &= F \cdot d \\ &= (m \cdot g) \cdot d \\ &= 1.2 \cdot 9.8 \cdot 0.7 \\ &= 8.232 \text{ J} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad W &= F \cdot d \\ &= 20 \cdot 6 \\ &= 120 \text{ ft-lb} \end{aligned}$$

When the force is constant, work is simply the product of force and distance: $W = F \cdot d$. If the force varies with position, we treat it as a function $F(x)$ defined on the interval $[a, b]$. How can we compute the work done as an object moves from $x = a$ to $x = b$?

- Approximate the work done over a small subinterval:

On $[x_{i-1}, x_i]$, force is roughly constant $\Rightarrow W_i \approx \text{force} \cdot \text{distance} = F(x_i^*) \cdot \Delta x$



- Approximate the total work over the entire interval $[a, b]$:

$$W_{\text{Total}} \approx \sum_{i=1}^n F(x_i^*) \Delta x$$

↑
↑
↑

Sum over all subintervals approximate force on each subinterval distance

- Calculate the exact total work for a variable force:

$$W_{\text{Total}} = \lim_{n \rightarrow \infty} \sum_{i=1}^n F(x_i^*) \Delta x = \int_a^b F(x) dx$$

↑
↑

make the intervals really small definition of integral (what an integral actually is)

Example. When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

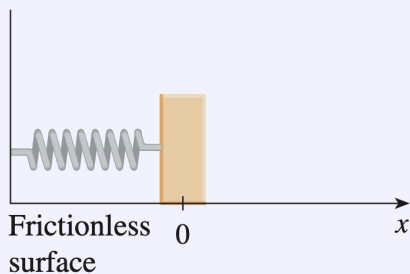
$$W = \int_a^b F(x) dx = \int_1^3 x^2 + 2x dx = \left[\frac{x^3}{3} + x^2 \right]_1^3 = \frac{50}{3} \text{ ft-lb}$$

Three types of problems: springs, cables, and tanks.

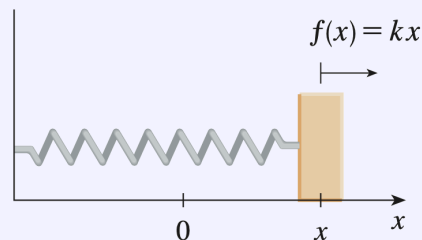
Theorem (Hooke's Law). The force required to maintain a spring stretched x units beyond its natural length is proportional to x :

$$f(x) = kx$$

where k is a positive constant called the spring constant. Hooke's Law holds provided that x is not too large.



(a) Natural position of spring



(b) Stretched position of spring

Example. A force of 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to a length of 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

$$W = \int_a^b F(x) dx = \int_a^b kx dx$$

Hooke's Law
↓

① Find k .

↙ Stretched distance is 5cm = 0.05 m

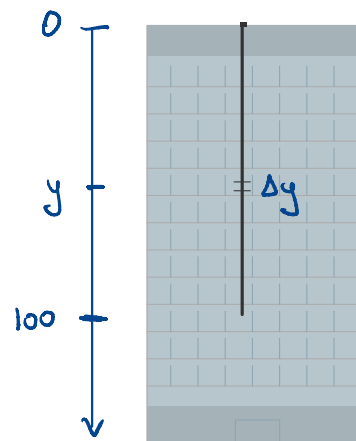
$$40 = k \cdot 0.05 \quad \Rightarrow \quad k = \frac{40}{0.05} = 800$$

② Find the interval $[a, b]$. We stretch the spring from 15cm = 0.15 m to 18cm = 0.18 m. Relative to the resting position 10cm = 0.10 m, x ranges from 0.05 m to 0.08 m

$$W = \int_{0.05}^{0.08} 800x dx = 800 \left[\frac{x^2}{2} \right]_{0.05}^{0.08} = 1.96 \text{ J}$$

Example. A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building.

- (a) How much work is required to lift the cable to the top of the building?
- (b) How much work is required to pull up only 20 feet of the cable?



(a) Consider a part of the cable of length Δy at height y . What is the work required to lift this slice to the top?

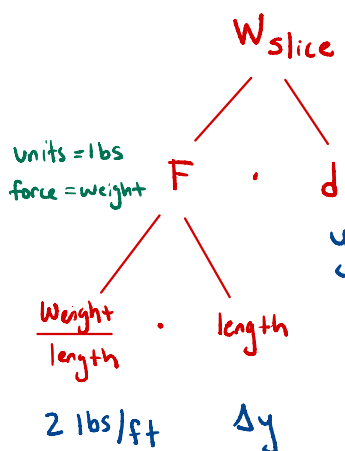
$$W_{\text{slice}} = 2 \cdot y \cdot \Delta y$$

Integrating over all pieces, we get

$$W_{\text{Total}} = \int_0^{100} 2y \, dy$$

The pieces are now infinitely small, so Δy became dy

$$= [y^2]_0^{100} = \boxed{10,000 \text{ ft-lb}}$$



(b) $W = W_1 + W_2$

Top part that moves a variable distance. $W_{\text{slice}} = 2y \cdot \Delta y$

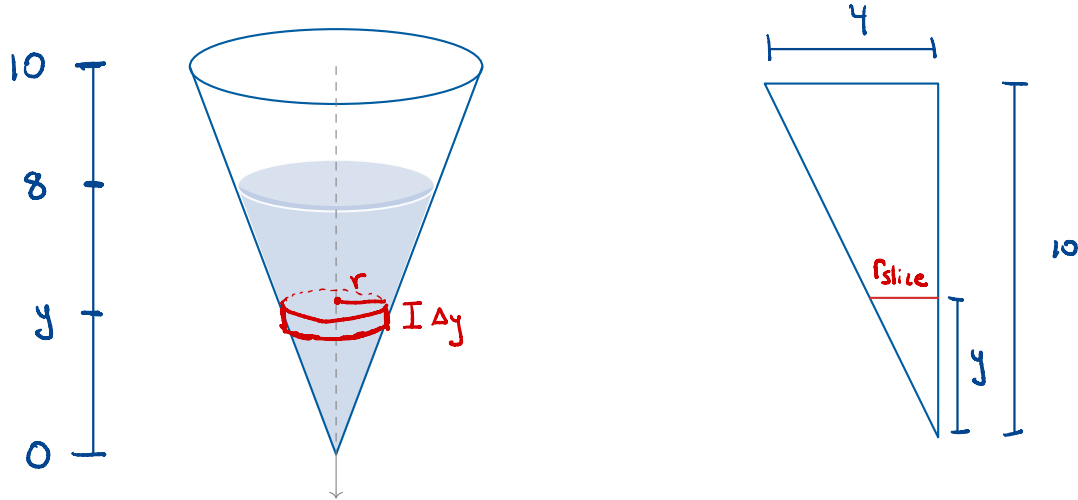
Bottom part that all moves 20 ft. $W_{\text{slice}} = 40 \cdot \Delta y$

$$W_1 = \int_0^{20} 2y \, dy = [y^2]_0^{20} = 400 \text{ ft-lb}$$

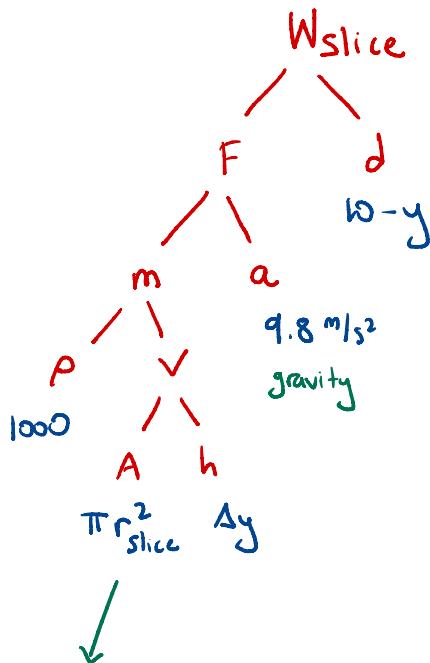
$$W_2 = \int_{20}^{100} 40 \, dy = 3200 \text{ ft-lb}$$

$$W_{\text{Total}} = \boxed{3600 \text{ ft-lb}}$$

Example. A tank has the shape of an inverted circular cone with a height of 10 m and a base radius of 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3 .)



Consider a slice of thickness Δy at height y . What is the work required to lift this slice?



$$W_{\text{slice}} = 1000 \pi \left(\frac{2}{5}y\right)^2 \cdot 9.8 \cdot (10-y) \Delta y$$

$$W_{\text{total}} = \int_0^8 9800 \pi \left(\frac{2}{5}y\right)^2 \cdot (10-y) dy$$

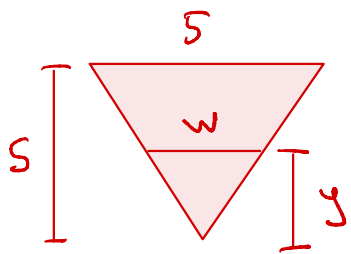
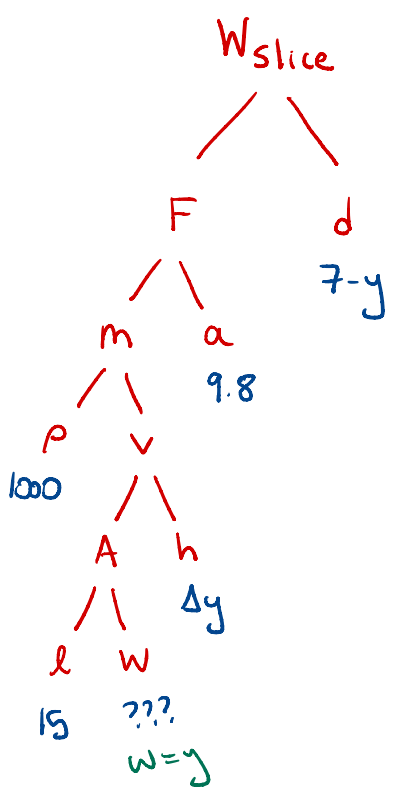
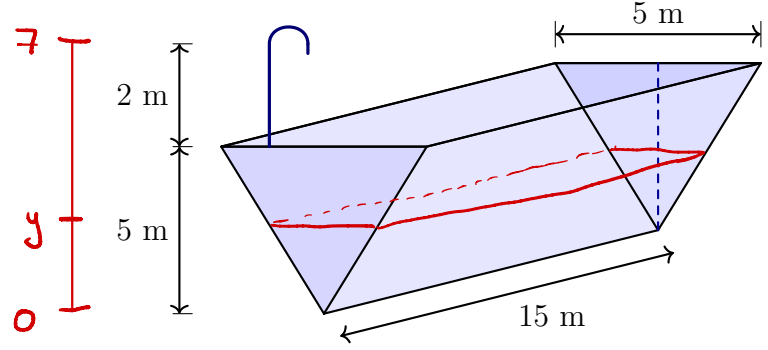
$$\approx 3,362,827.8 \text{ J}$$

Always integrate over the water!!!
This is where the slices are!!!

By similar triangles, $\frac{r_{\text{slice}}}{y} = \frac{4}{10} \Rightarrow r_{\text{slice}} = \frac{2}{5}y$

WebAssign

Problem. A tank is full of water. Find the work W required to pump the water out of the spout. Use 9.8 m/s^2 for g . Use 1000 kg/m^3 as the density of water.



By similar triangles,
 $\frac{5}{S} = \frac{w}{y} \Rightarrow w = y$

$$W_{\text{slice}} = 1000 \cdot 9.8 \cdot 15y \cdot (7-y) \cdot \Delta y$$

$$W_{\text{Total}} = \int_0^5 1000 \cdot 9.8 \cdot (15y) \cdot (7-y) dy$$

$\rho \cdot g \cdot \underbrace{A(y)} \cdot \underbrace{d(y)}$
 Slices between 0 and 5

In general, $W = \int_a^b \rho \cdot g \cdot A(y) \cdot d(y) dy$