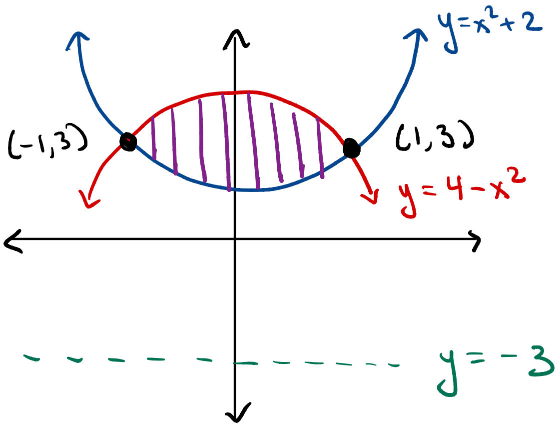
Warmup

Formula: $\int_a^b \pi R^2 - \pi r^2 \, dx$

Problem. Find the volume of the solid obtained by rotating the region bounded by

$$y = x^2 + 2 \qquad \text{and} \qquad y = 4 - x^2$$

about the horizontal line $y = -3$.



Intersection Points:

$x^2 + 2 = 4 - x^2$

$2x^2 = 2$

$x^2 = 1$

$x = 1$ or $x = -1$

washers stacked horizontally ⇝ use dx

$$V = \int_a^b \underbrace{\pi R(x)^2}_{\text{outer circle}} - \underbrace{\pi r(x)^2}_{\text{hole}} \, dx$$

$$= \int_{-1}^{1} \pi \left(4 - x^2 + 3\right)^2 - \pi \left(x^2 + 2 + 3\right)^2 \, dx$$

$$= \int_{-1}^{1} \pi (7 - x^2)^2 - \pi (x^2 + 5)^2 \, dx$$

$$= \pi \int_{-1}^{1} (49 - 14x^2 + x^4) - (x^4 + 10x^2 + 25) \, dx$$

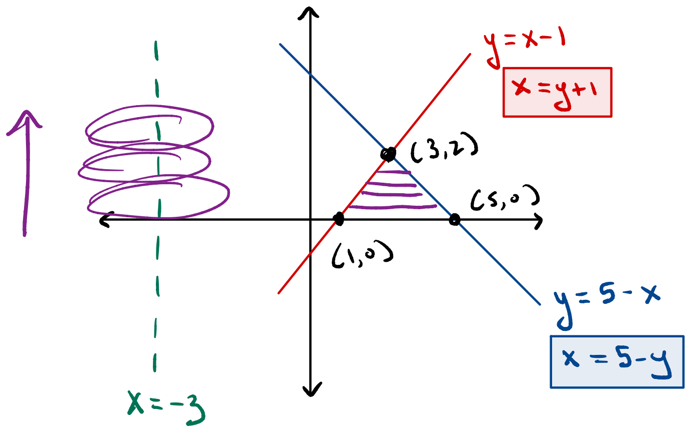
$$= \pi \int_{-1}^{1} 24 - 24x^2 \, dx$$

$$= \pi \left[24x - 24\frac{x^3}{3} \right]_{-1}^{1} = 32\pi$$

Problem. Find the volume of the solid obtained by rotating the region bounded by

$$y = x - 1, \qquad y = 5 - x, \qquad \text{and} \qquad y = 0$$

about the vertical line $x = -3$.



Intersection Points:

$$x - 1 = 5 - x$$

$$2x = 6$$

$$x = 3$$

Washers stacked vertically $\rightsquigarrow$ use dy

$$V = \int_c^d \underbrace{\pi R(y)^2}_{\text{outer circle}} - \underbrace{\pi r(y)^2}_{\text{hole}} \, dy$$

$$= \int_0^2 \pi (5 - y + 3)^2 - \pi (y + 1 + 3)^2 \, dy$$

$$= \pi \int_0^2 (8 - y)^2 - (y + 4)^2 \, dy$$

$$= 48\pi$$

Draw a "slice" in the 2D region you are rotating

perpendicular to axis of rotation → disks & washers

parallel to axis of rotation → shell

6.3 Volumes by Cylindrical Shells

The **cylindrical shell method** uses slices *parallel to the axis of rotation*. A thin slice revolves into a *cylindrical shell*, and we add up the shells to get the total volume.

$$V = 2\pi \int (\text{radius})(\text{height})(\text{thickness}).$$

If the axis of rotation is vertical, use vertical slices:

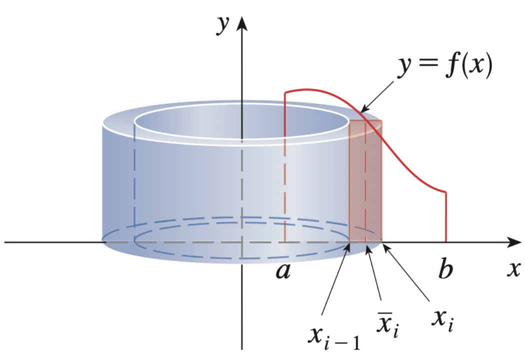
- *Thickness:* dx
- *Radius:* distance from x to the axis $x = a$: $r = |x - a|$
- *Height:* vertical length of the region at that x

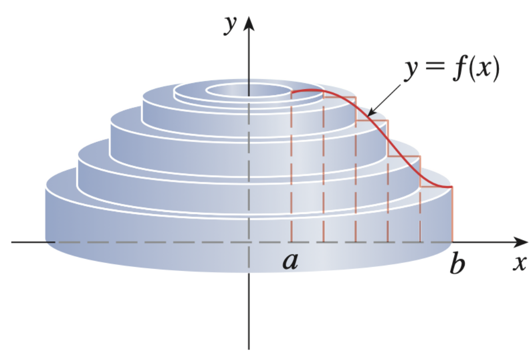
$$V = 2\pi \int_{x=a}^{x=b} r(x)\, h(x)\, dx.$$

If the axis of rotation is horizontal, use horizontal slices:

- *Thickness:* dy
- *Radius:* distance from y to the axis $y = b$: $r = |y - b|$
- *Height:* horizontal length of the region at that y

$$V = 2\pi \int_{y=c}^{y=d} r(y)\, h(y)\, dy.$$





A vertical line with thickness Δx produces a shell

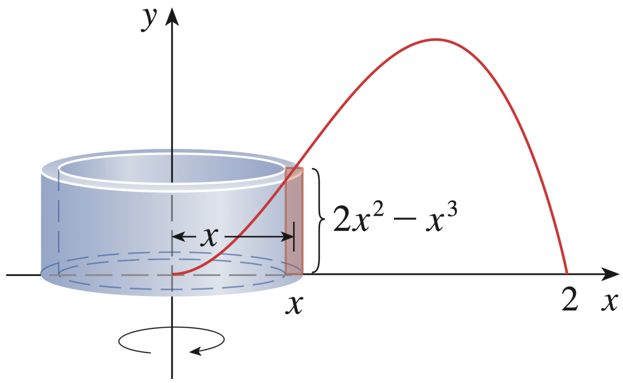
V = circumference * height * thickness

Total Volume of Solid:

$$V = \int_a^b \underbrace{2\pi r(x)}_{\text{circumference of the shell at } x} \cdot \underbrace{h(x)}_{\text{height of shell at } x} \cdot \underbrace{dx}_{\text{thickness of shell}}$$

integrate over all cylinders

Example. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.



Note: can't use washers because r(x) and R(x) are the same curve...

$$V = \int_0^2 \underbrace{2\pi x}_{\text{circumference}} \cdot \underbrace{(2x^2 - x^3)}_{\text{height}} \cdot \underbrace{dx}_{\text{thickness}}$$

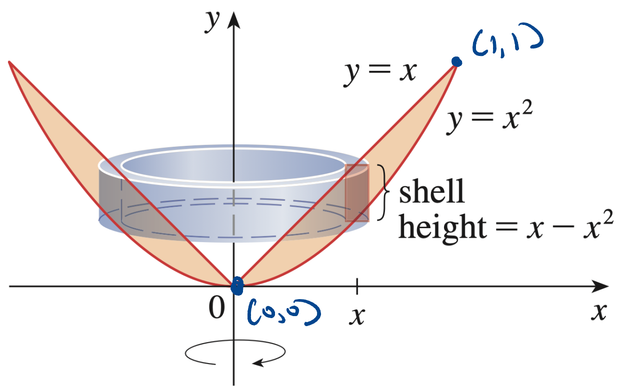
$$= 2\pi \int_0^2 (2x^3 - x^4)\, dx$$

$$= 2\pi \left[\frac{1}{2}x^4 - \frac{1}{5}x^5 \right]_0^2$$

$$= 2\pi \left[8 - \frac{32}{5} \right]$$

$$= \frac{16\pi}{5}$$

Example. Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.



$$V = \int_0^1 \underbrace{2\pi x}_{\text{circumference}} \cdot \underbrace{(x - x^2)}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

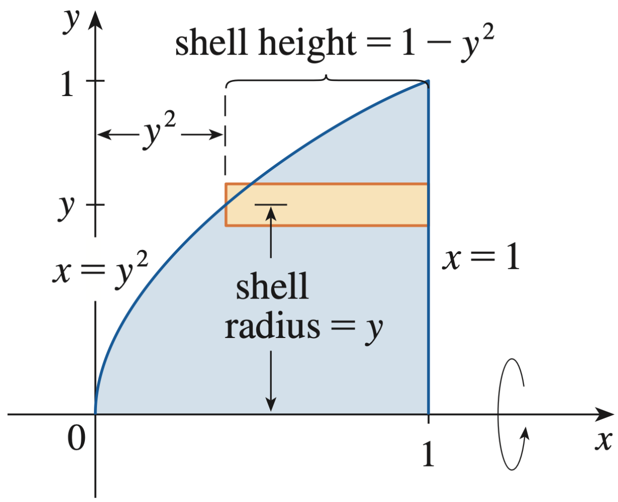
$$= 2\pi \int_0^1 x^2 - x^3 \, dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{\pi}{6}$$

Example. Use cylindrical shells to find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

* We did this in 6.2 using disks, which is probably easier.



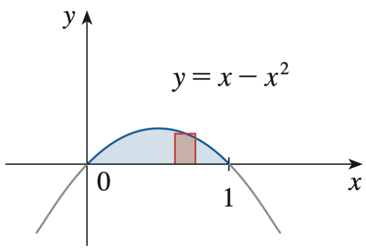
$$V = \int_0^1 \underbrace{2\pi y}_{\text{circumference}} \cdot \underbrace{(1-y^2)}_{\text{height}} \cdot \underbrace{dy}_{\text{thickness}}$$

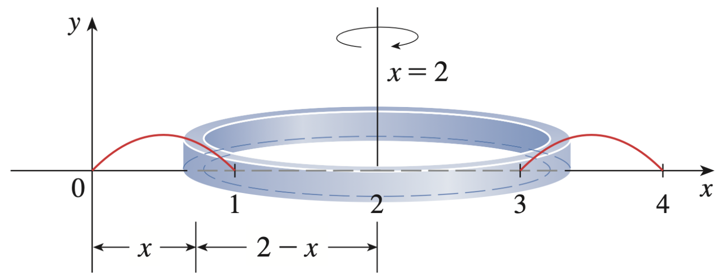
$$= 2\pi \int_0^1 y - y^3 \, dy$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1$$

$$= \frac{\pi}{2}$$

Example. Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.





$$V = \int_0^1 \underbrace{2\pi(2-x)}_{\text{circumference}} \cdot \underbrace{(x-x^2)}_{\text{height}} \; \underbrace{dx}_{\text{thickness}}$$

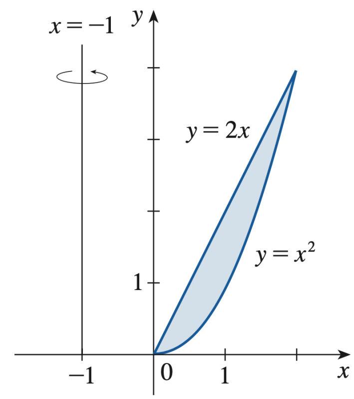
Integrate over the x-coords in the 2D region

$$= 2\pi \int_0^1 x^3 - 3x^2 + 2x \, dx$$

$$= 2\pi \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1$$

$$= \frac{\pi}{2}$$

Example. Consider the region in the first quadrant bounded by the curves $y = x^2$ and $y = 2x$. A solid is formed by rotating the region about the line $x = -1$.



Find the volume of the solid using:

(a) x as the variable of integration.

(b) y as the variable of integration.

Solution: