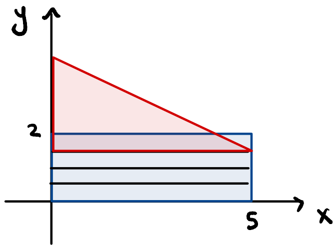
Warm-up:

4. A solid has a rectangular base in the xy-plane: $0 \leq x \leq 5$, $0 \leq y \leq 2$. Cross-sections perpendicular to the y-axis are right triangles whose legs lie along the base plane. Specifically, for each fixed y, the base of the right triangle is 5 (the full length in the x direction) and the height of the triangle is y. Find the volume of the solid.



$$A(y) = \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 5 \cdot y = \frac{5}{2} y$$

$$V = \int_0^2 \frac{5}{2} y \, dy = \frac{5}{2} \cdot \left[\frac{y^2}{2} \right]_0^2 = \frac{5}{2} \cdot \frac{4}{2} = 5$$

6.2 Volumes of Solids of Revolution

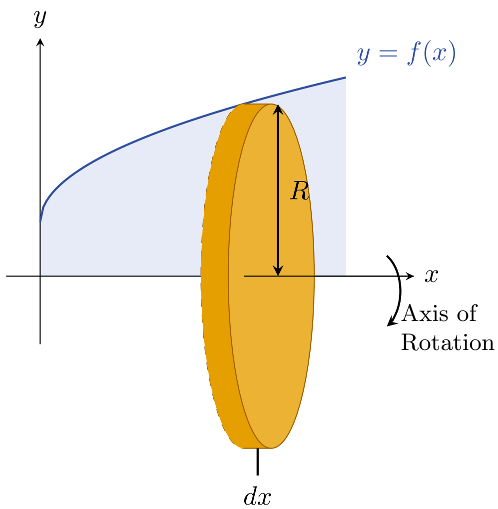
When a region is revolved around an axis, we can compute the volume by adding up thin circular cross-sections.

1. Disk method: cross-sections are *solid* circles.

horizontal axis of rotation

$$V = \pi \int_a^b \left(R(x)\right)^2 dx \qquad \text{or} \qquad V = \pi \int_c^d \left(R(y)\right)^2 dy$$

vertical axis of rotation



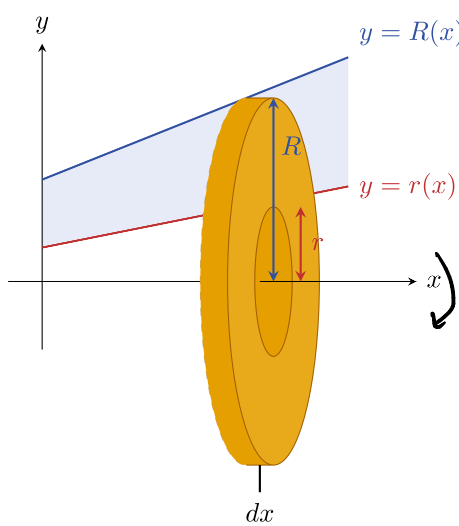
At each x, we get a disk with area

$$\pi \left[R(x)\right]^2$$

2. Washer method: cross-sections have a *hole*.

$$V = \pi \int_a^b \left(\left(R(x)\right)^2 - \left(r(x)\right)^2\right) dx \qquad \text{or} \qquad V = \pi \int_c^d \left(\left(R(y)\right)^2 - \left(r(y)\right)^2\right) dy$$

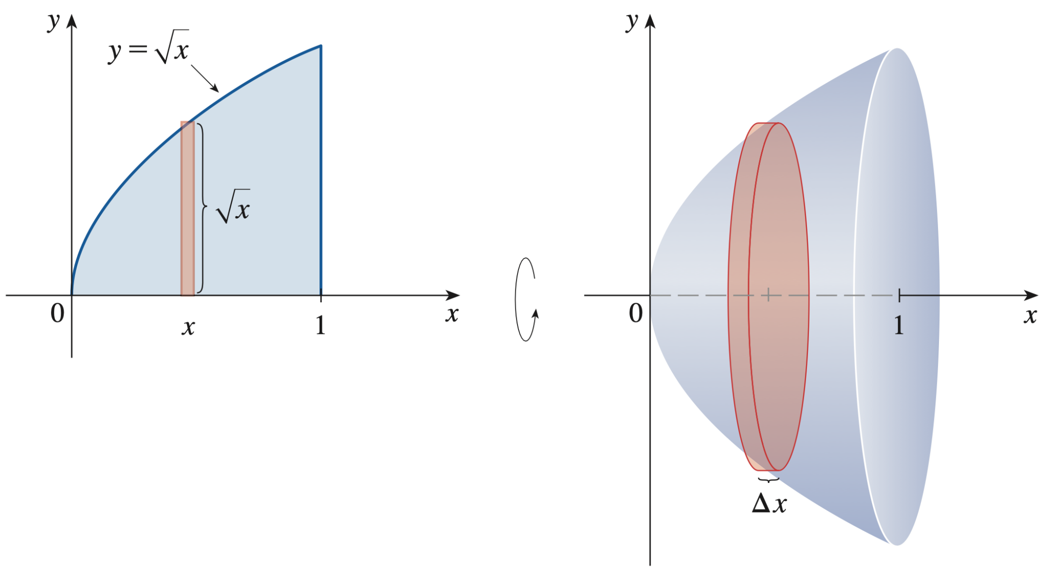
If there is space between the region and the axis of rotation, we get a washer with



axis of rotation

$$\underbrace{\pi R(x)^2}_{\text{outer disk}} - \underbrace{\pi r(x)^2}_{\text{hole}}$$

Example. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

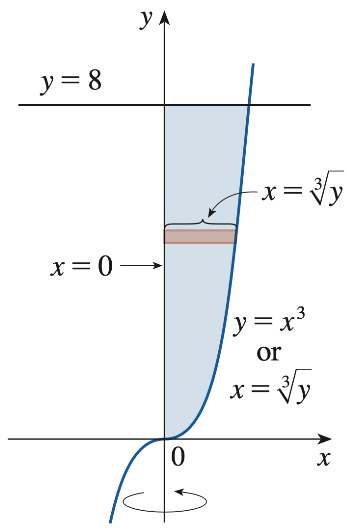


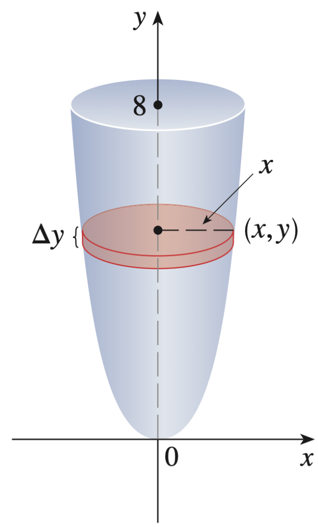
The slice at x is a disk with radius $\sqrt{x}$

The area of the cross-section is $A(x) = \pi \underbrace{(\sqrt{x})}_{\text{radius}}{}^2 = \pi x$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x \, dx = \pi \left[\frac{x^2}{2} \right]_0^1 = \frac{\pi}{2}$$

Example. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.





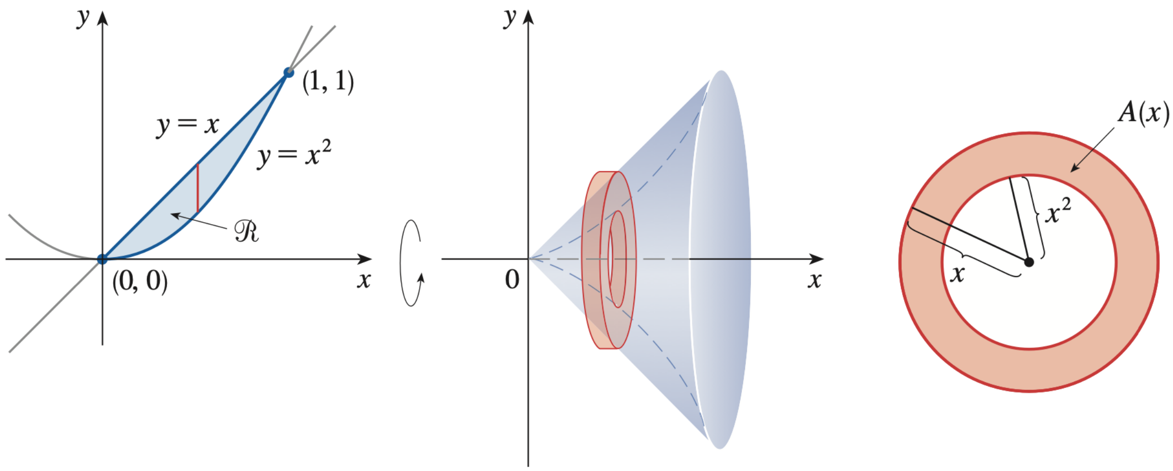
The region is being rotated about the y-axis $\Rightarrow$ slice the solid perpendicular to the y-axis and integrate with respect to y.

At height y, the disk has radius $x = \sqrt[3]{y}$

The area of the disk is $A(y) = \pi \underbrace{(\sqrt[3]{y})}_{\text{radius}}^2 = \pi y^{2/3}$

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \pi \left[\frac{3}{5} y^{5/3} \right]_0^8 = \frac{96\pi}{5}$$

Example. The region R enclosed by the curves $y = x$ and $y = x^2$ is rotated about the x-axis. Find the volume of the resulting solid.



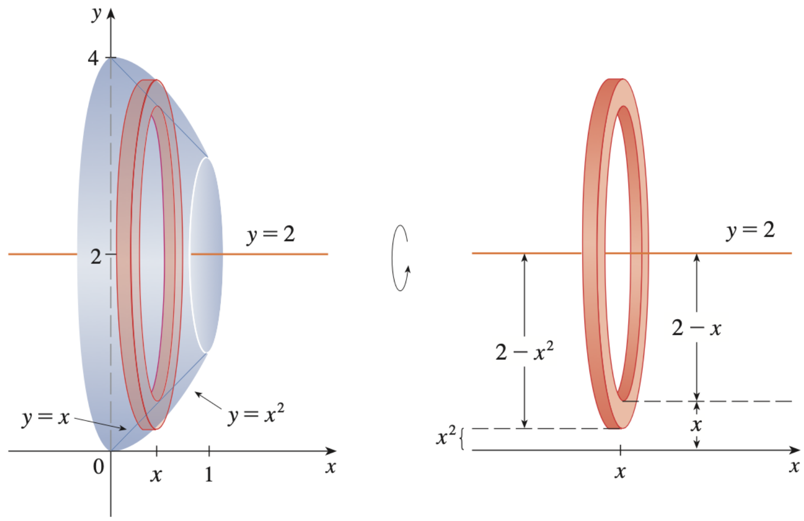
At each x-coordinate, the cross-sectional area is

$$A(x) = \pi \underbrace{(x)}_{\text{outer radius}}{}^2 - \pi \underbrace{(x^2)}_{\text{inner radius}}{}^2 = \pi\left(x^2 - x^4\right)$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left(x^2 - x^4\right) dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{2\pi}{15}$$

Example. Find the volume of the solid obtained by rotating the region R enclosed by the curves $y = x$ and $y = x^2$ about the line $y = 2$.



The cross-sectional area is $A(x) = \pi [R(x)]^2 - \pi [r(x)]^2$

$$= \pi [\underbrace{2-x^2}_{\text{outer radius}}]^2 - \pi [\underbrace{2-x}_{\text{inner radius}}]^2$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi \left[(2-x^2)^2 - (2-x)^2 \right] dx$$

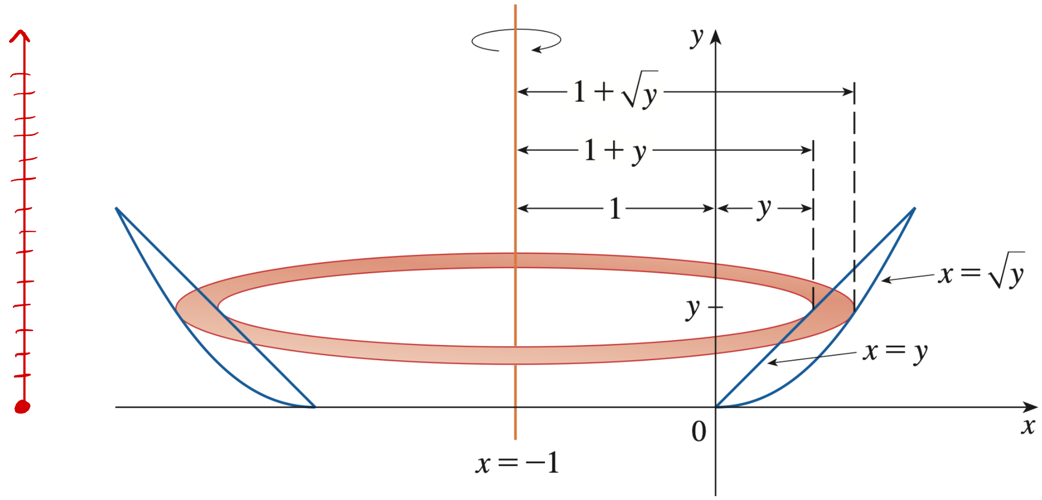
$$= \pi \int_0^1 x^4 - 5x^2 + 4x \, dx$$

$$= \pi \left[\frac{x^5}{5} - 5 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} \right]_0^1$$

$$= \frac{8\pi}{15}$$

Example. Find the volume of the solid obtained by rotating the region R enclosed by the curves $y = x$ and $y = x^2$ about the line $x = -1$.

The washers are stacked vertically
⇓
integrate with respect to y



The cross-sectional area is

$$A(y) = \text{outer circle} - \text{hole}$$

$$= \pi\Big[\underbrace{1+\sqrt{y}}_{\text{outer radius}}\Big]^2 - \pi\Big[\underbrace{1+y}_{\text{inner radius}}\Big]^2$$

$$V = \int_0^1 A(y)\,dy = \pi\int_0^1 (1+\sqrt{y})^2 - (1+y)^2\,dy$$

$$= \pi\int_0^1 2\sqrt{y} - y - y^2\,dy$$

$$= \pi\left[\frac{4}{3}y^{3/2} - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{\pi}{2}$$