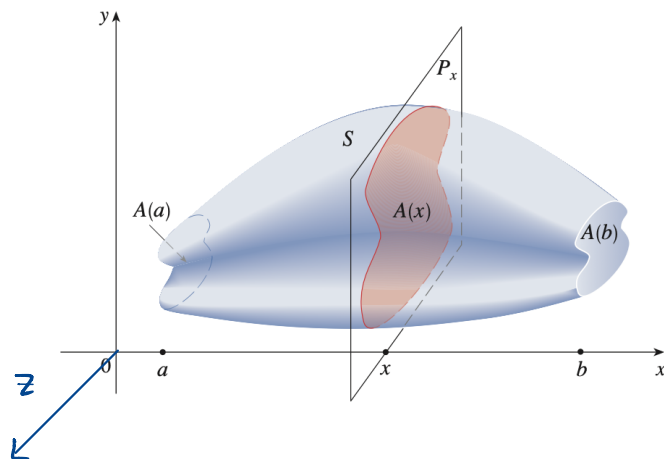
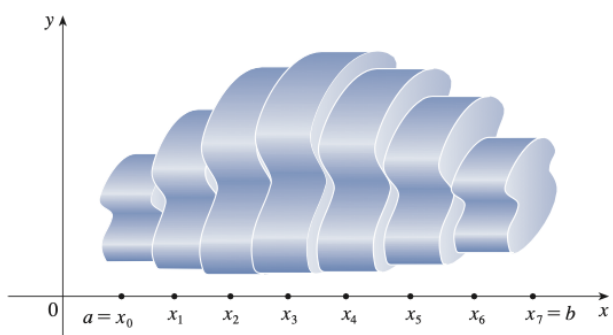
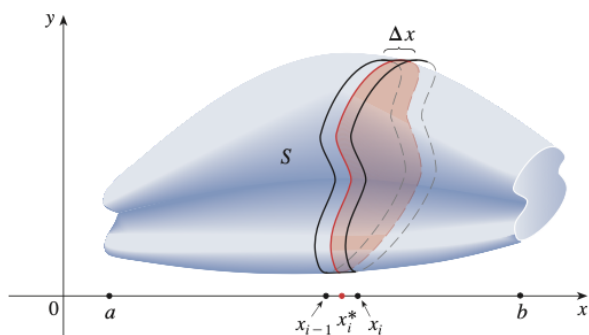


6.2 Volumes by Known Cross-Sections

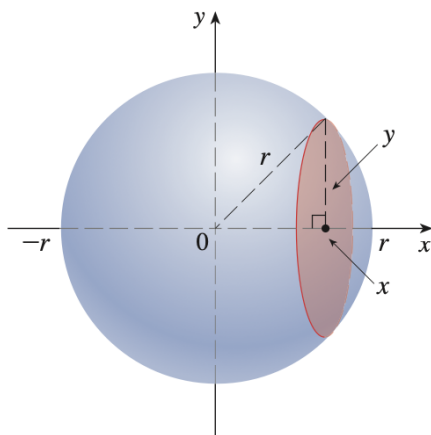
Question. How can we find the volume of a solid region S ?



- Let $A(x)$ be the area of the cross-section in the plane P_x , which is the plane perpendicular to the x -axis, passing through x .
(think about slicing S with a knife at x)
- The area $A(x)$ varies as x increases from a to b
- The total volume of S is $\int_a^b A(x) dx$

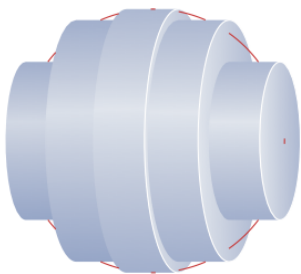


Example. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

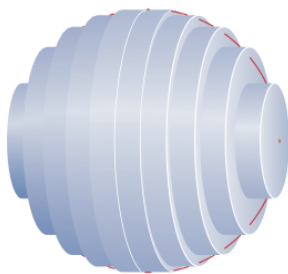


- The plane P_x intersects the sphere in a circle $y = \sqrt{r^2 - x^2}$
- The cross-sectional area is $A(x) = \pi y^2 = \pi (r^2 - x^2)$

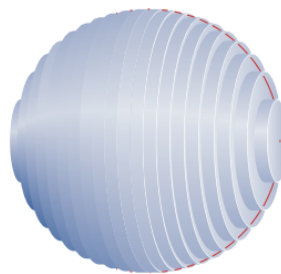
$$\begin{aligned}
 V &= \int_{-r}^r A(x) \, dx = \int_{-r}^r \pi (r^2 - x^2) \, dx = \pi \left[r^2 x - \frac{x^3}{3} \right]_{-r}^r \\
 &= \pi \left[\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right] \\
 &= \pi \left[r^3 + r^3 - \frac{r^3}{3} - \frac{r^3}{3} \right] \\
 &= \frac{4}{3} \pi r^3
 \end{aligned}$$



(a) Using 5 disks, $V \approx 4.2726$



(b) Using 10 disks, $V \approx 4.2097$

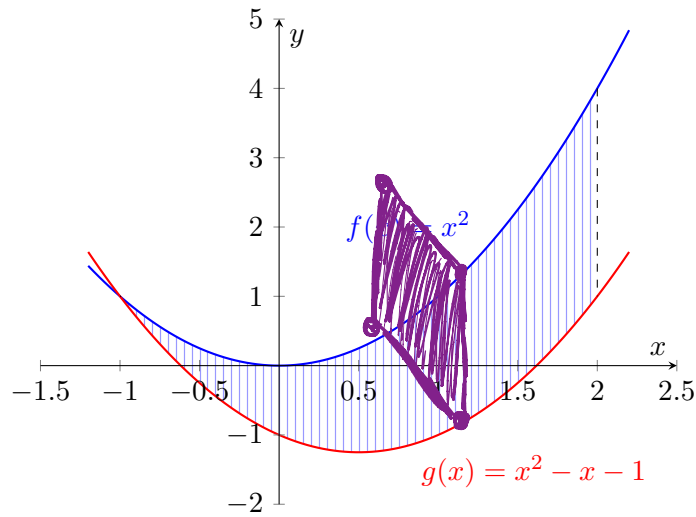


(c) Using 20 disks, $V \approx 4.1940$

Approximating the volume of a sphere with radius 1

Exam-style

Example. Let R be the region bounded by $f(x) = x^2$ and $g(x) = x^2 - x - 1$ on the interval $[-1, 2]$. A solid has base R , and cross-sections perpendicular to the x -axis are squares. Find the volume of the solid.



Side of the square at x

$$\begin{aligned} \textcircled{1} A(x) &= [s(x)]^2 = [f(x) - g(x)]^2 = [x^2 - (x^2 - x - 1)]^2 \\ &= [x + 1]^2 \end{aligned}$$

$$\begin{aligned} \textcircled{2} V &= \int_{-1}^2 A(x) dx = \int_{-1}^2 (x+1)^2 dx \\ &= \int_0^3 u^2 dx \\ &= \left[\frac{1}{3} u^3 \right]_0^3 \\ &= 9 \end{aligned}$$

let $u = x + 1$
 $du = dx$

Other cross-sections

- squares
- rectangles
- triangles
- semi/quarter-circles