5.5 The Substitution Rule

Theorem. Let $u = g(x)$ be differentiable on an interval and suppose f is continuous on the range of g. Then

$$\int f(g(x))\, g'(x)\, dx \;=\; \int f(u)\, du.$$

In particular, if you see an integrand that looks like $f(g(x)) \cdot g'(x)$, you can simplify the integral by making the substitution

$$u = g(x) \qquad \text{and} \qquad du = g'(x)\, dx.$$

Note: the symbols dx and du are called *differentials*. For a deeper discussion of differentials, see Section 3.10.

Example. Find $\displaystyle\int x^3 \cos(x^4 + 2)\; dx$.

Example. Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Example. Calculate $\displaystyle\int \tan x\,dx$.

Example. Calculate $\int e^{5x}\ dx$.

Example. Calculate $\int \frac{e^{1/x}}{x^2}\ dx$.

Example. Calculate $\int x\sqrt{x-1}\,dx$

Example. Calculate $\int x^5\sqrt{1+x^2}\,dx$.

Theorem (The Substitution Rule for Definite Integrals)**.** Let $u = g(x)$ be differentiable on $[a, b]$ and assume g' is continuous on $[a, b]$. If f is continuous on the range of g, then

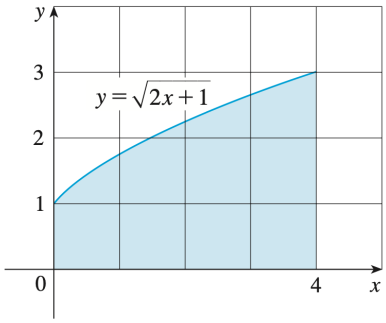
$$\int_a^b f(g(x))\, g'(x)\, dx \;=\; \int_{g(a)}^{g(b)} f(u)\, du.$$

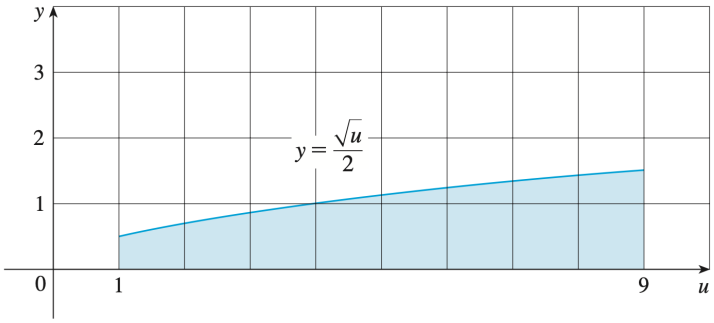
How to use it. Set $u = g(x)$, so $du = g'(x)\, dx$, *and change the bounds*:

$$x = a \;\Rightarrow\; u = g(a), \qquad x = b \;\Rightarrow\; u = g(b).$$

Then rewrite the entire integral in u-language and evaluate.

Example. Evaluate $\displaystyle\int_0^4 \sqrt{2x+1}\, dx$.





Example. Evaluate $\int_0^2 xe^{x^2}\,dx$.

Example. Calculate $\int_1^e \frac{\ln x}{x}\,dx$.