5.5 The Substitution Rule → Reverse the chain rule

Theorem. Let $u = g(x)$ be differentiable on an interval and suppose f is continuous on the range of g. Then

$$\int f(g(x))\, g'(x)\, dx \;=\; \int f(u)\, du.$$

In particular, if you see an integrand that looks like $f(g(x)) \cdot g'(x)$, you can simplify the integral by making the substitution

$$u = g(x) \qquad \text{and} \qquad du = g'(x)\, dx.$$

Note: the symbols dx and du are called *differentials*. For a deeper discussion of differentials, see Section 3.10.

Example. Find $\int x^3 \cos(x^4 + 2)\, dx$.

Let $u = x^4 + 2$. Then $du = 4x^3\, dx \Rightarrow x^3\, dx = \frac{1}{4} du$

Hence $\int x^3 \cos(x^4+2)\, dx = \int \cos(u) \cdot \frac{1}{4} du$

$$= \frac{1}{4} \sin(u) + C$$

$$= \frac{1}{4} \sin(x^4+2) + C$$

Example. Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\, dx$.

Let $u = 1-4x^2$. Then $du = -8x \cdot dx \Rightarrow x \cdot dx = -\frac{1}{8}\, du$

$$\int \frac{x}{\sqrt{1-4x^2}}\, dx = \int \frac{1}{\sqrt{u}} \cdot -\frac{1}{8} du = -\frac{1}{8}\int \frac{1}{\sqrt{u}}\, du$$

$$= -\frac{1}{8} \cdot \int u^{-1/2} du$$

$$= -\frac{1}{8} \cdot 2u^{1/2} + C$$

$$= -\frac{1}{4} u^{1/2} + C$$

$$= -\frac{1}{4}(1-4x^2)^{1/2} + C$$

Example. Calculate $\displaystyle\int \tan x\, dx$.

- Write $\int \tan x\, dx$ as $\displaystyle\int \frac{\sin x}{\cos x}\, dx$
- Let $u = \cos x \Rightarrow du = -\sin x \cdot dx$

  $\Rightarrow -du = \sin x \cdot dx$

- Hence $\displaystyle\int \tan x\, dx = \int \frac{1}{\cos x} \cdot \sin x\, dx = \int \frac{1}{u} \cdot -du$

$$= -\int \frac{1}{u}\, du$$

$$= -\ln|u| + C$$

$$= -\ln(|\cos x|) + C$$

Example. Calculate $\int e^{5x}\,dx$.

Let $u = 5x$. Then $du = 5\,dx \Rightarrow dx = \frac{1}{5}du$

$$\int e^{5x}\,dx = \int e^u \cdot \tfrac{1}{5}du = \frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C$$

$$= \frac{1}{5}e^{5x} + C$$

Example. Calculate $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$.

Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}\cdot dx \Rightarrow -du = \frac{1}{x^2}dx$

$$\int \frac{e^{1/x}}{x^2}\,dx = \int e^u \cdot -du = -\int e^u\,du = -e^u + C$$

$$= -e^{1/x} + C$$

Example. Calculate $\int x\sqrt{x-1}\,dx$

Let $u = x-1$. Then $du = dx$

Have an extra x factor. Note: $x = u+1$

$$\int x\sqrt{x-1}\,dx = \int (u+1)\cdot\sqrt{u}\,du$$

$$= \int u^{3/2} + u^{1/2}\,du$$

$$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$

Example. Calculate $\int x^5\sqrt{1+x^2}\,dx$. $= \int x^4\cdot x\cdot\sqrt{1+x^2}\,dx$

Let $u = 1+x^2$. Then $du = 2x\cdot dx \Rightarrow x\cdot dx = \frac{1}{2}du$

Have an extra x^4 factor. Note: $x^4 = (x^2)^2 = (u-1)^2$

$$\int x^5\cdot\sqrt{1+x^2}\,dx = \int (u-1)^2\cdot\sqrt{u}\cdot\frac{1}{2}du$$

$$= \frac{1}{2}\int (u^2-2u+1)\cdot\sqrt{u}\,du$$

$$= \frac{1}{2}\int u^{5/2} - 2u^{3/2} + u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2} - 2\cdot\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$$

Theorem (The Substitution Rule for Definite Integrals)**.** Let $u = g(x)$ be differentiable on $[a, b]$ and assume g' is continuous on $[a, b]$. If f is continuous on the range of g, then

$$\int_a^b f(g(x))\, g'(x)\, dx \;=\; \int_{g(a)}^{g(b)} f(u)\, du.$$

How to use it. Set $u = g(x)$, so $du = g'(x)\, dx$, *and change the bounds*:

$$x = a \;\Rightarrow\; u = g(a), \qquad x = b \;\Rightarrow\; u = g(b).$$

Then rewrite the entire integral in u-language and evaluate.

Example. Evaluate $\displaystyle\int_0^4 \sqrt{2x+1}\, dx$.

Let $u = 2x+1$. Then $du = 2\,dx \Rightarrow dx = \frac{1}{2}du$

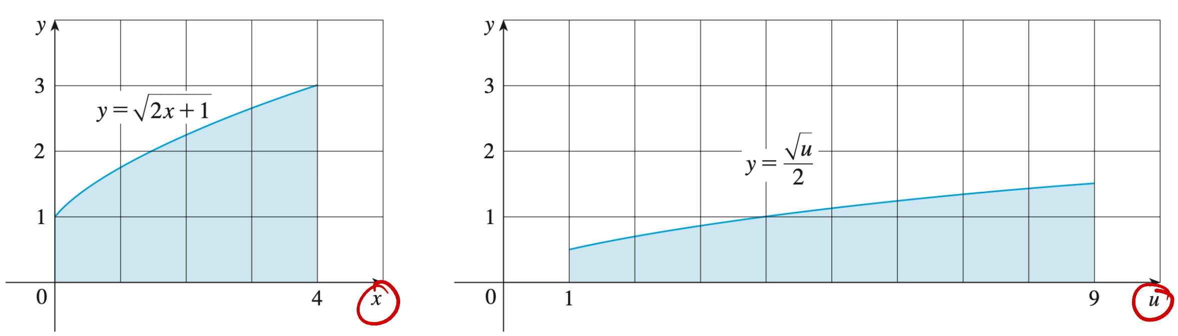
Change bounds:

When $x = 0$: $u = 2\cdot 0 + 1 = 1$
When $x = 4$: $u = 2\cdot 4 + 1 = 9$

$$\int_{x=0}^{x=4} \sqrt{2x+1}\, dx = \int_{u=1}^{u=9} \sqrt{u}\cdot \frac{1}{2}du = \left[\frac{1}{2}\cdot\frac{2}{3}u^{3/2}\right]_{u=1}^{u=9}$$

$$= \left[\frac{1}{3}u^{3/2}\right]_{u=1}^{u=9}$$

$$= \frac{1}{3}\cdot 9^{3/2} - \frac{1}{3}\cdot 1^{3/2} = \boxed{\frac{26}{3}}$$



These areas are the same!

Example. Evaluate $\int_0^2 xe^{x^2}\,dx$.

Let $u = x^2$. Then $du = 2x \cdot dx \Rightarrow x \cdot dx = \frac{1}{2}du$

When $x=0$: $u=0$

when $x=2$: $u=4$

$$\int_{x=0}^{x=2} x \cdot e^{x^2}\,dx = \int_{u=0}^{u=4} e^u \cdot \frac{1}{2}\,du = \left[\frac{1}{2}e^u\right]_{u=0}^{u=4}$$

$$= \frac{1}{2}e^4 - \frac{1}{2}e^0$$

$$= \frac{e^4}{2} - \frac{1}{2}$$

Note: For definite integrals, we don't need to back-substitute, we just get a final answer.

Example. Calculate $\int_1^e \frac{\ln x}{x}\,dx$.

Let $u = \ln(x)$. Then $du = \frac{1}{x}\,dx$

When $x=1$: $u = \ln(1) = 0$

when $x=e$: $u = \ln(e) = 1$

$$\int_{x=1}^{x=e} \frac{\ln(x)}{x}\,dx = \int_{u=0}^{u=1} u\,du = \left[\frac{1}{2}u^2\right]_{u=0}^{u=1} = \frac{1}{2}$$