often a p-series or a geometric series

11.4 The Comparison Tests

Comparison tests determine convergence or divergence by comparing a given series to one whose behavior is already known. The Direct Comparison Test compares terms directly, while the Limit Comparison Test examines the limit of the ratio of corresponding terms.

The Direct Comparison Test

Theorem. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is also convergent.
2. If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is also divergent.

Example. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges. $\longrightarrow$ Looks like $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2}$, which converges

Introduction: Let $a_n = \dfrac{5}{2n^2+4n+3}$ and $b_n = \dfrac{5}{2n^2}$.

Both are positive, so the D.C.T. applies.

Apply the Test: We have $\dfrac{5}{2n^2+4n+3} \le \dfrac{5}{2n^2}$ for $n \ge 1$.

bigger denominator

$\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$.

Conclusion: Since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges and $a_n \le b_n$ for $n \ge 1$, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges by the D.C.T.

The Limit Comparison Test

Theorem. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Example. Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence. → This looks like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ which converges

⚠ The D.C.T. doesn't work with $b_n = \frac{1}{2^n}$

The inequality $\frac{1}{2^n-1} > \frac{1}{2^n}$ is useless!

(bigger than "converges" means nothing)

Intro: Let $a_n = \frac{1}{2^n-1}$ and $b_n = \frac{1}{2^n}$. Both are positive, so LCT applies.

Apply LCT:

- $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/2^n-1}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n-1} \cdot \frac{1/2^n}{1/2^n} = \lim_{n\to\infty} \frac{1}{1-1/2^n} = \frac{1}{1-0} = 1$

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series since $|r| = \frac{1}{2} < 1$

Conclusion: Since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$,

$\displaystyle\sum_{n=1}^{\infty} a_n$ converges by the L.C.T.