Linear Approximation

1. Let $f(x) = \ln(1 + 2x)$ and $a = 0$.
   (a) Find the linearization $L(x)$ at $x = 0$.

   Solution: $L(x) = f'(0)(x - 0) + f(0)$.

   $f(0) = \ln(1) = 0$

   $f'(x) = \dfrac{2}{1 + 2x}$. So $f'(0) = 2$.

   $\boxed{L(x) = 2x}$

   (b) Use $L(x)$ to approximate $\ln(1.06)$. State the x you plug into L.

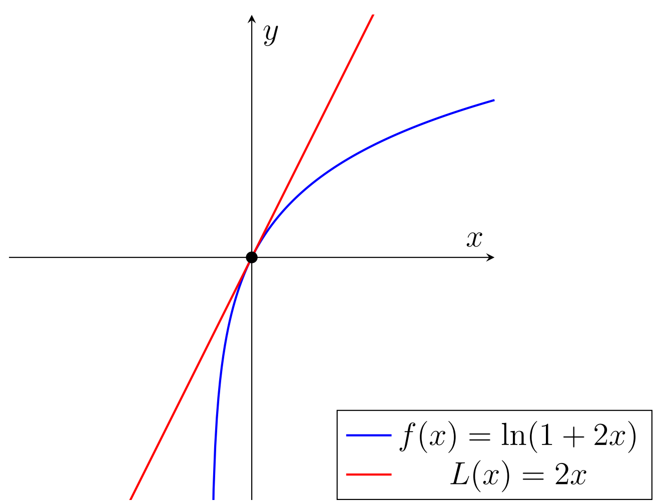
   Solution: Set $1 + 2x = 1.06$. Then $x = 0.03$.

   $$\ln(1.06) = f(0.03) \approx L(0.03) = 2(0.03) = \boxed{0.06}$$

   (c) Is $L(x)$ an overestimate or an underestimate?

   Solution:

   $$f''(x) = -\frac{4}{(1 + 2x)^2} < 0 \Rightarrow \text{concave down} \Rightarrow \boxed{\text{overestimate}}$$



2. Let $f(x) = \sqrt[3]{x}$ and $a = 8$.

   (a) Find $L(x)$ at $a = 8$.

   Solution: $L(x) = f'(8)(x-8) + f(8)$.

   $f(8) = 2$

   $f'(x) = \frac{1}{3}x^{-2/3}$. So $f'(8) = \frac{1}{12}$

   $$L(x) = 2 + \frac{1}{12}(x-8)$$

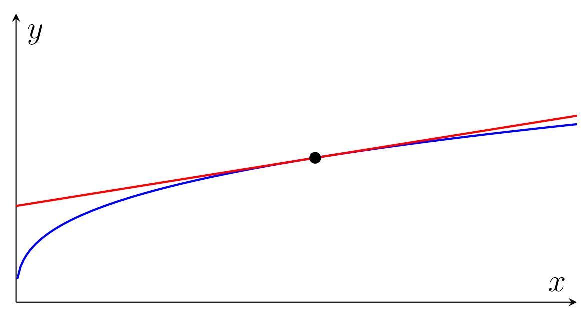
   (b) Use L to approximate $\sqrt[3]{7.7}$ and $\sqrt[3]{8.2}$. Are these approximations overestimates or underestimates?

   Solution:

   $$\sqrt[3]{7.7} = f(7.7) \approx L(7.7) = 2 + \frac{1}{12} \cdot \frac{-3}{10} = \frac{240}{120} - \frac{3}{120} = \boxed{\frac{237}{120}}$$

   $$\sqrt[3]{8.2} = f(8.2) \approx L(8.2) = 2 + \frac{1}{12} \cdot \frac{2}{10} = 2 + \frac{1}{60} = \boxed{\frac{121}{60}}$$

   Since $f''(x) = -\frac{2}{9}x^{-5/3} < 0$ for $x > 0$, both are *overestimates*.



3. Let $f(x) = \dfrac{1}{x^2+5}$ and $a = 1$.

(a) Find $L(x)$ at $x = 1$.

Solution: $L(x) = f'(1)(x-1) + f(1)$.

$$f(1) = \frac{1}{6}$$

$$f'(x) = -\frac{2x}{(x^2+5)^2}. \text{ So } f'(1) = -\frac{1}{18}.$$

$$\boxed{L(x) = \frac{1}{6} - \frac{1}{18}(x-1)}.$$

(b) Use L to approximate $f(1.1)$ and $f(0.9)$ and decide whether L is an overestimate or an underestimate near $a = 1$.

Solution:

- $L(1.1) = \dfrac{1}{6} - \dfrac{1}{18} \cdot \dfrac{1}{10} = \dfrac{30}{180} - \dfrac{1}{180} = \boxed{\dfrac{29}{180}}$
- $L(0.9) = \dfrac{1}{6} + \dfrac{1}{18} \cdot \dfrac{1}{10} = \dfrac{30}{180} + \dfrac{1}{180} = \boxed{\dfrac{31}{180}}$

Here $f''(1) = \dfrac{-4}{216} < 0$ (concave down), so *overestimates near* 1.

4. Let $f(x) = \sqrt{1+x}$ and $a = 0$.

(a) Find $L(x)$.

Solution:

$$f(0) = 1$$

$$f'(x) = \frac{1}{2\sqrt{1+x}}. \text{ So } f'(0) = \frac{1}{2}.$$

$$\boxed{L(x) = 1 + \frac{1}{2}x}$$

(b) Use L to estimate $\sqrt{1.04}$ and decide if your estimate is an overestimate or an underestimate.

Solution:

$$\sqrt{1.04} = f(0.04) \approx L(0.04) = 1 + \tfrac{1}{2}(0.04) = \boxed{1.02}.$$

Since $f''(x) = -\frac{1}{4}(1+x)^{-3/2} < 0$, it's *an overestimate.*

5. Below are four functions and four linearizations at $a = 0$. Match each function with its correct $L(x)$ *and* justify each match by computing $f(0)$ and $f'(0)$.

(I) $f(x) = \sqrt{4+x}$ at $a = 0$

(II) $f(x) = \dfrac{1}{3+x}$ at $a = 0$

(III) $f(x) = \sin(2x)$ at $a = 0$

(IV) $f(x) = e^x$ at $a = 0$

(A) $L(x) = 2x$

(B) $L(x) = 1 + x$

(C) $L(x) = 2 + \dfrac{1}{4}x$

(D) $L(x) = \dfrac{1}{3} - \dfrac{1}{9}x$

Solution:

- **(I)** $\to$ **(C).** $f(0) = 2$, $f'(x) = \dfrac{1}{2\sqrt{4+x}} \Rightarrow f'(0) = \frac{1}{4}$. So $\boxed{L(x) = 2 + \frac{1}{4}x.}$

- **(II)** $\to$ **(D)**. $f(0) = \dfrac{1}{3}$, $\quad f'(0) = -\dfrac{1}{9}$. So $\boxed{L(x) = \frac{1}{3} - \frac{1}{9}x.}$

- **(III)** $\to$ **(A).** $f(0) = \sin 0 = 0$, $f'(x) = 2\cos(2x) \Rightarrow f'(0) = 2$. So $\boxed{L(x) = 2x.}$

- **(IV)** $\to$ **(B).** $f(0) = 1$, $f'(0) = 1$. $\boxed{L(x) = 1 + x.}$