

Final Exam Study Guide

MATH1300 - Calculus I

Fall 2025

Contents

Final Exam Information	1
5.1 - Areas and Distances	3
5.2 - The Definite Integral	5
5.3 - The Fundamental Theorem of Calculus, Part 1	7
5.4 - FTC Part 2 and Indefinite Integrals	9
5.5 - The Substitution Rule	11
6.1 - Areas Between Curves	12

Final Exam Information

The final exam will be held on **Monday, December 8** from **7:30 am to 10:00 am**. It is **comprehensive**, covering all topics listed below.

1.1–1.3 Precalculus Review Algebra and function basics needed for calculus: lines, quadratics, exponentials, function notation, and graphing.

2.1 The Tangent & Velocity Problems The original motivating problems for derivatives: slopes of tangent lines and instantaneous velocity as limits of secant slopes and average velocities.

2.2 Limits The basic limit concept: understanding $\lim_{x \rightarrow a} f(x)$, interpreting limits from graphs and tables, and recognizing when limits do or do not exist.

2.3 Limit Laws Algebraic rules for computing limits (sum, product, quotient, power, etc.) and using them to evaluate limits exactly.

2.5 Continuity & IVT Continuity at a point and on an interval, types of discontinuities, and the Intermediate Value Theorem (and how to use it to show an equation has a root).

2.6 Limits at Infinity & Horizontal Asymptotes Behavior of functions as $x \rightarrow \pm\infty$, limits of rational functions, and identifying horizontal asymptotes from limits at infinity.

2.7 Derivatives & Rates of Change The derivative as a limit of difference quotients, basic interpretation as a rate of change, and first derivative notation.

4.3 What f' and f'' say about f How the first derivative controls increasing/decreasing behavior and local extrema, and how the second derivative describes concavity and points of inflection.

3.1 Derivatives of Polynomials and Exponential Functions Basic derivative rules: power rule, constant multiple rule, sum rule, and derivatives of e^x and other exponential functions.

3.2 Product and Quotient Rules Differentiation formulas for products and quotients of functions, and practice applying them in combination with earlier rules.

3.3 Derivatives of Trig Functions Derivatives of $\sin x$, $\cos x$, and the other basic trigonometric functions, including using chain/product/quotient rules with trig.

3.4 The Chain Rule Differentiating compositions $f(g(x))$; inside–outside thinking and rewriting derivatives using both Leibniz and prime notation.

3.5 Implicit Differentiation Differentiating relationships involving x and y without solving for y explicitly, including finding $\frac{dy}{dx}$ and related slopes.

3.9 Related Rates Word problems involving quantities that change over time, using the chain rule and implicit differentiation to relate their rates.

3.6 Derivatives of Inverse Trigonometric Functions Formulas and applications for derivatives of inverse trig functions such as $\arcsin x$, $\arctan x$, etc.

3.6 Derivatives of Logarithmic Functions Logarithmic differentiation and derivatives of $\ln x$ and related log expressions (including using logs to differentiate complicated products and powers).

- 3.7 Rates of Change in the Natural and Social Sciences** Interpreting derivative formulas in context: position–velocity–acceleration, growth and decay, and other applied rate-of-change models.
- 3.10 Linear Approximation** Using tangent lines to approximate function values near a point (linearization) and understanding differentials in this context.
- 4.1 Maximum and Minimum Values** Absolute and local extrema, critical numbers, and using derivatives and endpoints to find max/min values on an interval.
- 4.2 The Mean Value Theorem** Statement and interpretation of the MVT, plus Rolle’s Theorem, and how they connect average and instantaneous rates of change.
- 4.3 & 4.5 Curve Sketching** Using f' and f'' (increasing/decreasing, concavity, inflection points, asymptotes, intercepts) to sketch and analyze graphs.
- 4.4 Indeterminate Forms and L’Hôpital’s Rule** Identifying indeterminate forms like $0/0$ and ∞/∞ , and using L’Hôpital’s Rule to evaluate such limits.
- 4.7 Applied Optimization** Setting up and solving optimization problems: modeling a quantity, finding a suitable function, and using derivatives to locate optimal values.
- 4.9 Antiderivatives** Basic indefinite integration as the reverse of differentiation; families of antiderivatives and the constant of integration $+C$.
- 5.1 Areas and Distances** Approximating area under a curve using Riemann sums and interpreting definite integrals as limits of these sums.
- 5.2 The Definite Integral** The definite integral as signed area, properties of integrals, and computing integrals from graphs and simple formulas.
- 5.3 FTC Part 1** The Fundamental Theorem of Calculus, Part 1: defining accumulation functions $F(x) = \int_a^x f(t) dt$ and relating their derivatives to f .
- 5.4 FTC Part 2, Indefinite Integrals** The Fundamental Theorem of Calculus, Part 2: evaluating definite integrals using antiderivatives, and connecting definite and indefinite integrals.
- 5.5 The Substitution Rule** Integration by substitution as the reverse of the chain rule; choosing a substitution and changing variables (and limits) correctly.
- 6.1 Areas Between Curves** Computing areas of regions bounded by multiple curves using $\int(\text{top} - \text{bottom}) dx$ or $\int(\text{right} - \text{left}) dy$.

5.1 - Areas and Distances

For a continuous function f on $[a, b]$, we can approximate the area under its graph using *Riemann sums*, where the interval $[a, b]$ is divided into n equal subintervals of width $\Delta x = \frac{b-a}{n}$.

• **Left sum:** $L_n = \sum_{i=0}^{n-1} f(x_i) \Delta x$ (use *left endpoints*).

• **Right sum:** $R_n = \sum_{i=1}^n f(x_i) \Delta x$, (use *right endpoints*).

• **Midpoint sum:** $M_n = \sum_{i=1}^n f(m_i) \Delta x$, where m_i is the midpoint of the i -th subinterval.

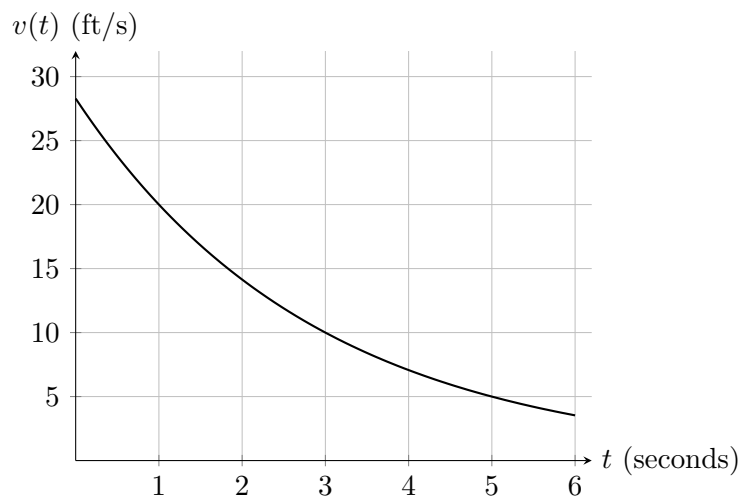
If f is increasing, then L_n is a *lower* estimate and R_n is an *upper* estimate. If f is decreasing, the roles reverse. Midpoint sums M_n are often more accurate than L_n or R_n for the same n .

- Let $f(x) = 2 + x^2$ on the interval $[0, 2]$.
 - Use *right endpoints* to find R_4
 - Use *left endpoints* to find L_4
 - Use *midpoints* to find M_4
 - Based on the fact that f is increasing on $[0, 2]$, decide which of L_4, M_4, R_4 is a lower estimate, which is an upper estimate, and which seems most accurate.
- The speed $v(t)$ (in ft/s) of a runner increases steadily during the first 3 seconds of a race. Her speed is recorded every half-second as shown in the table.

t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
$v(t)$ (ft/s)	0.0	5.8	9.1	13.7	16.4	18.9	20.0

- Use a sum with $\Delta t = 0.5$ and *left endpoints* to find a *lower estimate* for the distance she travels during the first three seconds.
- Use a sum with $\Delta t = 0.5$ and *right endpoints* to find an *upper estimate* for the distance she travels during the first three seconds.

3. The graph below shows the velocity $v(t)$ (in ft/s) of a braking car for $0 \leq t \leq 6$ seconds.



Use a *midpoint sum* with three equal subintervals to estimate the distance the car travels while the brakes are applied.

4. Let $f(x) = 3 + \sin x$ on the interval $[0, \pi]$. Use a Riemann sum with *right endpoints* to express the area under the graph of f from 0 to π as a limit.
5. Determine a region whose area is equal to the limit

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(2 + \frac{5i}{n} \right) \frac{3}{n}.$$

(Write your answer in the form “the area under $y = f(x)$ on $[a, b]$ ”.)

5.2 - The Definite Integral

For a continuous function f , the definite integral

$$\int_a^b f(x) dx$$

represents the *net signed area* between the graph of $y = f(x)$ and the x -axis from $x = a$ to $x = b$. Areas above the axis count positively and areas below the axis count negatively. We can define the integral as a limit of Riemann sums:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x, \quad \Delta x = \frac{b-a}{n},$$

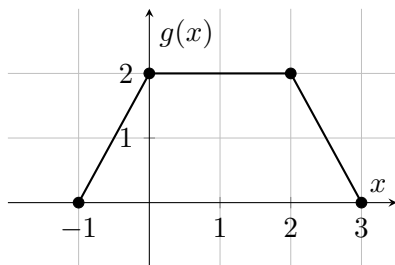
where x_i^* is a sample point in the i -th subinterval. Basic properties include:

$$\int_a^b (c f(x) + g(x)) dx = c \int_a^b f(x) dx + \int_a^b g(x) dx,$$

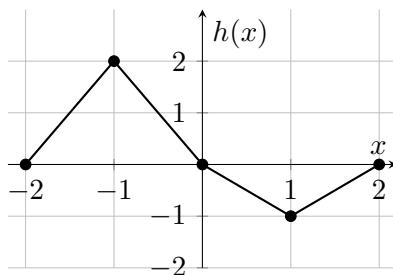
$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \quad \text{for } a < c < b,$$

$$\text{if } f(x) \leq g(x), \text{ then } \int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

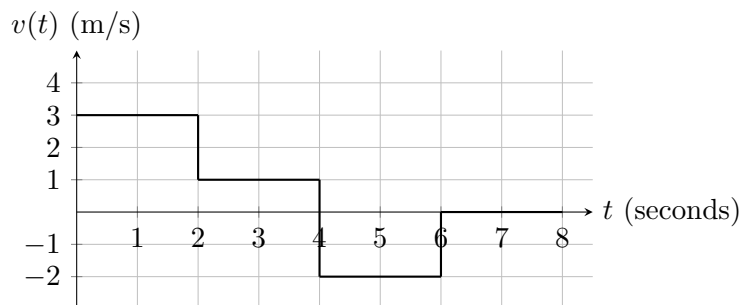
1. Use the graph of $y = g(x)$ to evaluate $\int_{-1}^3 g(x) dx$.



2. Use the graph of $y = h(x)$ to answer the following questions.



- (a) Compute the net signed area $\int_{-2}^2 h(x) dx$.
- (b) Compute the *total area* between the graph of $h(x)$ and the x -axis on $[-2, 2]$.
3. The graph below shows the velocity $v(t)$ (in meters per second) of a particle moving along a line, for $0 \leq t \leq 8$ seconds. The graph consists of horizontal line segments.



- (a) Find the *displacement* of the particle on the time interval $[0, 8]$.
- (b) Find the *total distance* traveled on $[0, 8]$.
4. Let $h(x)$ be a continuous function such that

$$\int_{-2}^5 h(x) dx = 11 \quad \text{and} \quad \int_1^5 h(x) dx = 7.$$

Find the value of $\int_{-2}^1 h(x) dx$.

5. Write the definite integral $\int_0^4 (1+3x) dx$ as the limit of right-hand Riemann sums using n subintervals of equal width on the interval $[0, 4]$.
6. Suppose $p(x) \leq 4$ for all x in the interval $[0, 5]$. What inequality *must* be satisfied by $\int_0^5 p(x) dx$? (Give a simple inequality involving a constant and the integral.)
7. The rate at which a machine produces widgets (in widgets per minute) is given by the function $r(t)$. Values of $r(t)$ are shown in the table below.

t (min)	0	2	4	6
$r(t)$ (widgets/min)	3	7	9	12

- Use a left-hand Riemann sum with three subintervals to approximate $\int_0^6 r(t) dt$.
- Assume that $r(t)$ is increasing and concave down on $[0, 6]$. Is the left-hand Riemann sum from part (a) an overestimate or an underestimate of the true value of the integral? Explain your reasoning.

5.3 - The Fundamental Theorem of Calculus, Part 1

Let f be continuous on $[a, b]$, and define

$$F(x) = \int_a^x f(t) dt.$$

Then F is differentiable on (a, b) and

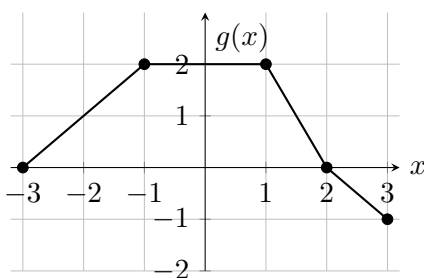
$$F'(x) = f(x).$$

More generally, if the upper limit is a differentiable function $g(x)$, then by the Chain Rule:

$$\frac{d}{dx} \left[\int_a^{g(x)} f(t) dt \right] = f(g(x)) g'(x).$$

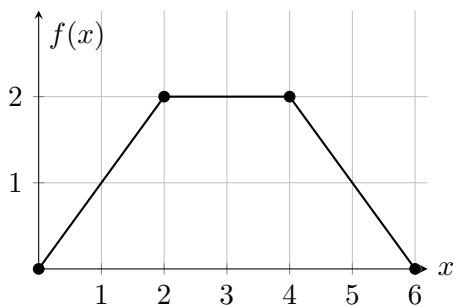
If the variable limit appears as the *lower* limit, we use $\int_x^a f(t) dt = - \int_a^x f(t) dt$.

1. Evaluate $\frac{d}{dx} \int_1^{x^2} \sqrt{1+t^3} dt$.
2. Evaluate $\frac{d}{dx} \int_0^{\sin x} (u^4 + 1) du$.
3. Evaluate $\frac{d}{dx} \int_x^5 (t^2 - 3t) dt$.
4. Let $G(x) = \int_{x^2}^{e^x} \cos(t^2) dt$. Find $G'(x)$.
5. Let $H(x) = \int_{-3}^x g(t) dt$, where the graph of $y = g(x)$ is shown below.



- (a) Use the Fundamental Theorem of Calculus to find a simple formula for $H'(x)$ in terms of g .
- (b) Use the graph to find the exact value of $H'(2)$.
- (c) Compute $H(1)$ by interpreting the integral as (signed) area.
- (d) On which intervals is $H(x)$ increasing? Explain in terms of the graph of $g(x)$.

6. The graph below shows the function $y = f(x)$ for $0 \leq x \leq 6$.



Define

$$F(x) = \int_0^x f(t) dt.$$

- (a) Compute $F(0)$, $F(2)$, $F(4)$, and $F(6)$ by interpreting each integral as an area under the graph of f .
- (b) Use the Fundamental Theorem of Calculus to write a formula for $F'(x)$ in terms of f .
- (c) On which intervals is $F(x)$ increasing? On which intervals is $F(x)$ decreasing? Justify using the graph of f .
- (d) At which value of x in $[0, 6]$ do you expect $F(x)$ to be largest? Explain briefly.
7. Let

$$F(x) = \int_0^x (t^2 - 4t + 3) dt.$$

- (a) Use the Fundamental Theorem of Calculus to find $F'(x)$ and $F''(x)$.
- (b) Find all critical numbers of F and classify each as a local maximum, local minimum, or neither.
- (c) Determine the intervals where $F(x)$ is increasing and where it is decreasing.

5.4 - FTC Part 2 and Indefinite Integrals

Fundamental Theorem of Calculus, Part 2. If F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

An *indefinite integral* $\int f(x) dx$ represents the entire family of antiderivatives of f , written $F(x) + C$, where C is an arbitrary constant. In practice, to evaluate a definite integral $\int_a^b f(x) dx$, we:

1. Find an antiderivative $F(x)$ of $f(x)$.
2. Compute $F(b) - F(a)$.

If we already know that a function G satisfies $G'(x) = f(x)$, then $\int_a^b f(x) dx = G(b) - G(a)$ without explicitly computing an indefinite integral.

1. Find the indefinite integral

$$\int (2e^x - 3x^2 + 4) dx.$$

2. Compute each indefinite integral.

(a) $\int (5x^4 - 7x^{-2} + 3x^{1/2}) dx$

(b) $\int (4 \cos x - 7 \sec^2 x) dx$

(c) $\int (e^{3x} + \frac{1}{x}) dx$ (assume $x > 0$).

3. Let k be differentiable for all real numbers and suppose that $k(-1) = 4$ and $k(3) = -2$. Evaluate

$$\int_{-1}^3 k'(x) dx.$$

4. Suppose F is an antiderivative of a function f and that $F(1) = 2$, $F(4) = 7$, and $F(-1) = 0$. Compute:

(a) $\int_1^4 f(x) dx$

(b) $\int_4^1 f(x) dx$

(c) $\int_{-1}^4 f(x) dx$

5. Evaluate each definite integral by finding an antiderivative and then using $F(b) - F(a)$.

(a) $\int_0^2 (x^2 - 4x + 5) dx$

(b) $\int_0^{\pi/2} (3 \cos x - 2 \sin x) dx$

(c) $\int_1^4 \left(\frac{2}{x} - \sqrt{x} \right) dx$

6. Let F be a function with derivative

$$F'(x) = 3x^2 - 1$$

and suppose $F(0) = 5$.

(a) Use an indefinite integral to find a formula for $F(x)$.

(b) Check your answer by differentiating.

(c) Use your formula to find $F(2)$ and verify that $F(2) - F(0) = \int_0^2 (3x^2 - 1) dx$.

7. A population $P(t)$ (in thousands of organisms) changes at a rate

$$P'(t) = 100 - 20t$$

(in thousands per year) for $0 \leq t \leq 4$. At time $t = 0$, the population is $P(0) = 500$ (thousand).

(a) Find the net change in the population from $t = 0$ to $t = 4$ by evaluating $\int_0^4 P'(t) dt$.

(b) Use your answer and the initial value $P(0) = 500$ to find $P(4)$.

(c) Explain in one sentence how this illustrates the Fundamental Theorem of Calculus, Part 2.

5.5 - The Substitution Rule

Substitution Rule (Indefinite Integrals). If $u = g(x)$ is differentiable and f is continuous on the range of g , then

$$\int f(g(x)) g'(x) dx = \int f(u) du,$$

where $du = g'(x) dx$.

In practice:

1. Choose $u = g(x)$ so that a complicated expression becomes simpler.
2. Compute $du = g'(x) dx$ and rewrite the integral entirely in terms of u .
3. Integrate with respect to u , then substitute back $u = g(x)$.

Substitution Rule (Definite Integrals). If $u = g(x)$ is differentiable on $[a, b]$, then

$$\int_a^b f(g(x)) g'(x) dx = \int_{u(a)}^{u(b)} f(u) du.$$

You may either change the limits to the new variable u (and not substitute back), or find an antiderivative in terms of u , substitute back to x , and then evaluate at $x = a$ and $x = b$.

1. Evaluate $\int \frac{3x}{x^2 + 1} dx$.
2. Evaluate $\int 4x \sqrt{x^2 + 9} dx$.
3. Evaluate $\int (2x - 1) e^{x^2 - x} dx$.
4. Evaluate $\int \cos(3x) \sin(3x) dx$.
5. Evaluate $\int \frac{1}{x \ln x} dx$, assuming $x > 1$.
6. Evaluate $\int_0^2 \frac{x}{x^2 + 1} dx$.
7. Evaluate $\int_0^{\frac{\pi}{4}} \cos^2 x \sin x dx$.
8. Evaluate $\int_1^4 \frac{x}{\sqrt{x^2 + 3}} dx$.
9. Evaluate $\int_0^1 x \sqrt{1 + x^2} dx$.

6.1 - Areas Between Curves

To find the area of the region between two curves:

- If functions are given as y -values on $[a, b]$ with $y_{\text{top}}(x) \geq y_{\text{bottom}}(x)$, then

$$A = \int_a^b (y_{\text{top}}(x) - y_{\text{bottom}}(x)) dx.$$

- If it is more natural to integrate with respect to y , write $x_{\text{right}}(y)$ and $x_{\text{left}}(y)$ and use

$$A = \int_c^d (x_{\text{right}}(y) - x_{\text{left}}(y)) dy.$$

- A region bounded by a single curve and an axis is a special case, with the axis playing the role of the “other curve”.

Always sketch the region and determine which curve is on top (or on the right) on each subinterval before setting up the integral. If the “top” and “bottom” curves change, split the integral at the intersection point(s).

1. Find the area of the region bounded by the curves

$$y = 4 - x^2 \quad \text{and} \quad y = -2x + 4.$$

2. Find the area of the region enclosed by the curves

$$y = x^2 \quad \text{and} \quad y = 2x + 3.$$

3. Find the area of the region bounded by

$$y = x^2 + 1 \quad \text{and} \quad y = 3x - 1.$$

4. The region is bounded by the curves

$$x = y^2 \quad \text{and} \quad x = 4 - y^2.$$

- (a) Sketch the region and decide whether it is easier to integrate with respect to x or y .
- (b) Set up and evaluate an integral (or integrals) that gives the area of this region.

5. The region R is bounded by the parabola $y = x^2$, the line $y = 4$, and the y -axis $x = 0$.

(a) Sketch the region and set up an integral of the form $\int_a^b (y_{\text{top}}(x) - y_{\text{bottom}}(x)) dx$ that represents the area of R . (You may need to split into more than one integral.)

(b) Set up an equivalent integral with respect to y , of the form $\int_c^d (x_{\text{right}}(y) - x_{\text{left}}(y)) dy$.

(c) Evaluate either of your integrals to find the area of R .

6. Let R be the region enclosed by the curves

$$y = |x| \quad \text{and} \quad y = 2.$$

(a) Sketch the region and find all intersection points.

(b) Express the area of R as a single definite integral, or as a sum of definite integrals, with respect to x .

(c) Evaluate to find the exact area.