Final Exam Study Guide – Solutions

MATH 1300 - Calculus I

Fall 2025

Contents

5.1 - Areas and Distances

1. For $f(x) = 2 + x^2$ on $[0, 2]$ with $n = 4$, we have

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$$

   (a) **Right endpoints.**

   Right endpoints: $x_i = 0 + i\Delta x = \frac{i}{2}$, $i = 1, \ldots, 4$.

$$\begin{aligned} R_4 &= \sum_{i=1}^{4} f(x_i)\,\Delta x = \frac{1}{2}\big(f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2)\big) \\ &= \frac{1}{2}\big((2 + (\tfrac{1}{2})^2) + (2 + 1^2) + (2 + (\tfrac{3}{2})^2) + (2 + 2^2)\big) \\ &= \frac{1}{2}(2.25 + 3 + 4.25 + 6) = \frac{31}{4} \approx 7.75. \end{aligned}$$

   (b) **Left endpoints.**

   Left endpoints: $x_i = 0 + (i-1)\Delta x = \frac{i-1}{2}$, $i = 1, \ldots, 4$.

$$\begin{aligned} L_4 &= \sum_{i=1}^{4} f(x_i)\,\Delta x = \frac{1}{2}\big(f(0) + f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2})\big) \\ &= \frac{1}{2}(2 + 2.25 + 3 + 4.25) = \frac{23}{4} \approx 5.75. \end{aligned}$$

   (c) **Midpoints.**

   Midpoints: $\bar{x}_i = 0 + \big(i - \frac{1}{2}\big)\Delta x = \frac{2i-1}{4}$, so $\bar{x}_1 = \frac{1}{4}$, $\bar{x}_2 = \frac{3}{4}$, $\bar{x}_3 = \frac{5}{4}$, $\bar{x}_4 = \frac{7}{4}$.

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(\bar{x}_i)\,\Delta x = \frac{1}{2}\big(f(\tfrac{1}{4}) + f(\tfrac{3}{4}) + f(\tfrac{5}{4}) + f(\tfrac{7}{4})\big) \\ &= \frac{1}{2}(2.0625 + 2.5625 + 3.5625 + 5.0625) = \frac{53}{8} \approx 6.625. \end{aligned}$$

   (d) **Lower vs. upper estimate.**

   Since f is increasing on $[0, 2]$,

$$L_4 < \int_0^2 f(x)\,dx < R_4,$$

   so L_4 is a *lower* estimate and R_4 is an *upper* estimate. The midpoint sum M_4 lies between them and is closest to the true value $\displaystyle\int_0^2 (2 + x^2)\,dx = \frac{20}{3} \approx 6.67$, so M_4 is the most accurate of the three.

2. We use $\Delta t = 0.5$ and the table of speeds.

   (a) **Lower estimate (left endpoints).**

   There are 6 subintervals on $[0, 3]$. Left endpoints: $t = 0, 0.5, 1.0, 1.5, 2.0, 2.5$.

$$\begin{aligned} \text{distance} &\approx \sum_{k=0}^{5} v(t_k)\,\Delta t \\ &= 0.5\,(0.0 + 5.8 + 9.1 + 13.7 + 16.4 + 18.9) \\ &= 0.5 \cdot 63.9 = 31.95 \text{ ft.} \end{aligned}$$

   (b) **Upper estimate (right endpoints).**

   Right endpoints: $t = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$.

$$\begin{aligned} \text{distance} &\approx \sum_{k=1}^{6} v(t_k)\,\Delta t \\ &= 0.5\,(5.8 + 9.1 + 13.7 + 16.4 + 18.9 + 20.0) \\ &= 0.5 \cdot 83.9 = 41.95 \text{ ft.} \end{aligned}$$

3. We divide $[0, 6]$ into three equal subintervals, so

$$\Delta t = \frac{6-0}{3} = 2.$$

   The midpoints are $t = 1, 3, 5$.

   From the graph we read approximate values

$$v(1) \approx 20, \quad v(3) \approx 10, \quad v(5) \approx 5 \text{ (ft/s).}$$

   The midpoint sum is

$$\begin{aligned} \text{distance} &\approx \Delta t\,[v(1) + v(3) + v(5)] \\ &= 2\,(20 + 10 + 5) = 70 \text{ ft.} \end{aligned}$$

4. On $[0, \pi]$ we divide into n equal pieces:

$$\Delta x = \frac{\pi - 0}{n} = \frac{\pi}{n}, \qquad x_i = 0 + i\Delta x = \frac{i\pi}{n} \quad (i = 1, \ldots, n),$$

   using right endpoints. Then

$$A = \int_0^{\pi} (3 + \sin x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \bigl(3 + \sin \tfrac{i\pi}{n}\bigr)\,\frac{\pi}{n}.$$

5. We rewrite

$$\sum_{i=1}^{n} \left(2 + \frac{5i}{n}\right) \frac{3}{n} = \sum_{i=1}^{n} \left(2 + \frac{5}{3} \cdot \frac{3i}{n}\right) \frac{3}{n}.$$

Here $\Delta x = \dfrac{3}{n}$ and $x_i = \dfrac{3i}{n}$ are the right endpoints in $[0,3]$. The sum is a Riemann sum for

$$f(x) = 2 + \frac{5}{3}x \quad \text{on} \quad [0,3].$$

Therefore the limit equals the area under $y = 2 + \frac{5}{3}x$ on $[0,3]$.

5.2 - The Definite Integral

1. From the graph of $y = g(x)$, the region from -1 to 3 is made of two right triangles and a rectangle.

$$\int_{-1}^{3} g(x)\,dx = \underbrace{\tfrac{1}{2}(1)(2)}_{\text{left triangle}} + \underbrace{(2)(2)}_{\text{rectangle}} + \underbrace{\tfrac{1}{2}(1)(2)}_{\text{right triangle}} = 1 + 4 + 1 = 6.$$

2. For $y = h(x)$:

   (a) On $[-2, 0]$ the graph is above the axis and forms a triangle with base 2 and height 2, so area $= \frac{1}{2} \cdot 2 \cdot 2 = 2$.

   On $[0, 2]$ the graph is below the axis and forms a triangle with base 2 and height 1, so signed area $= -\frac{1}{2} \cdot 2 \cdot 1 = -1$.

$$\int_{-2}^{2} h(x)\,dx = 2 + (-1) = 1.$$

   (b) Total area is the sum of absolute areas:

$$\text{total area} = 2 + 1 = 3.$$

3. For the velocity graph $v(t)$:

   (a) **Displacement**

$$\int_{0}^{8} v(t)\,dt = (3)(2) + (1)(2) + (-2)(2) + (0)(2) = 6 + 2 - 4 + 0 = 4 \text{ m}.$$

   (b) **Total distance**

$$\int_{0}^{8} |v(t)|\,dt = |3|(2) + |1|(2) + |-2|(2) + |0|(2) = 6 + 2 + 4 + 0 = 12 \text{ m}.$$

4. Use additivity of the integral:

$$\int_{-2}^{5} h(x)\,dx = \int_{-2}^{1} h(x)\,dx + \int_{1}^{5} h(x)\,dx.$$

   So

$$11 = \int_{-2}^{1} h(x)\,dx + 7 \quad \Rightarrow \quad \int_{-2}^{1} h(x)\,dx = 4.$$

5. On $[0,4]$ with n equal subintervals,

$$\Delta x = \frac{4-0}{n} = \frac{4}{n}, \qquad x_i = 0 + i\Delta x = \frac{4i}{n}, \quad i = 1, \ldots, n.$$

   The right-hand Riemann sum is

$$\sum_{i=1}^{n} (1 + 3x_i)\,\Delta x = \sum_{i=1}^{n} \left(1 + 3 \cdot \frac{4i}{n}\right) \frac{4}{n}.$$

   Thus

$$\int_0^4 (1+3x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{12i}{n}\right) \frac{4}{n}.$$

6. Since $p(x) \le 4$ on $[0,5]$, the largest the area under the graph can be is the area of the rectangle of height 4 and width 5:

$$\int_0^5 p(x)\,dx \;\le\; 4 \cdot (5-0) = 20.$$

7. For $[0,6]$ with three equal subintervals,

$$\Delta t = \frac{6-0}{3} = 2.$$

   Left endpoints: $t = 0, 2, 4$.

   - Approximation:

$$\int_0^6 r(t)\,dt \approx \sum_{\text{left}} r(t_k)\,\Delta t = 2\big(r(0) + r(2) + r(4)\big) = 2(3+7+9) = 38.$$

   - Since $r(t)$ is increasing, on each subinterval the left-endpoint rectangle lies below the graph, so the left-hand sum is an *underestimate* of the true integral.

5.3 - The Fundamental Theorem of Calculus, Part 1

1. Let

$$y = \int_1^u \sqrt{1+t^3}\,dt \qquad \text{and} \qquad u = x^2$$

By the Fundamental Theorem of Calculus and the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \sqrt{1+(u)^3}\cdot 2x = \sqrt{1+x^6}\cdot 2x.$$

2. Let

$$y = \int_0^v (u^4+1)\,du \qquad \text{and} \qquad v = \sin x$$

By the Fundamental Theorem of Calculus and the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx} = \big(v^4+1\big)\cdot\cos x = \big(\sin^4 x+1\big)\cdot\cos x.$$

3. Let

$$H(x) = \int_x^5 (t^2-3t)\,dt.$$

First rewrite with constant lower limit:

$$H(x) = -\int_5^x (t^2-3t)\,dt.$$

Now apply the Fundamental Theorem of Calculus:

$$H'(x) = -\big(x^2-3x\big) = -x^2+3x.$$

4. Let

$$G(x) = \int_{x^2}^{e^x} \cos(t^2)\,dt.$$

We first rewrite the integral so that each piece has a *constant* lower limit. Choose any fixed constant a (for example, $a=0$). Then

$$\begin{aligned}\int_{x^2}^{e^x}\cos(t^2)\,dt &= \int_a^{e^x}\cos(t^2)\,dt + \int_{x^2}^{a}\cos(t^2)\,dt\\ &= \int_a^{e^x}\cos(t^2)\,dt - \int_a^{x^2}\cos(t^2)\,dt.\end{aligned}$$

Now differentiate each term using the Fundamental Theorem of Calculus (Part 1) plus the Chain Rule.

$$G'(x) = \cos(e^{2x})\,e^x - 2x\cos(x^4).$$

5. We have

$$H(x) = \int_{-3}^{x} g(t)\,dt.$$

(a) By the Fundamental Theorem of Calculus,

$$H'(x) = g(x).$$

(b) From the graph, $g(2) = 0$. Hence

$$H'(2) = g(2) = 0.$$

(c)

$$H(1) = \int_{-3}^{1} g(t)\,dt.$$

From the graph:

- On $[-3, -1]$ we have a triangle of base 2 and height 2, so area $= \frac{1}{2} \cdot 2 \cdot 2 = 2$.
- On $[-1, 1]$ we have a rectangle of width 2 and height 2, so area $= 2 \cdot 2 = 4$.

All of this region lies above the x-axis, so the integral equals the total area:

$$H(1) = 2 + 4 = 6.$$

(d) Since $H'(x) = g(x)$, the function H is increasing where $g(x) > 0$ and decreasing where $g(x) < 0$. From the graph, $g(x) > 0$ for $-3 < x < 2$ and $g(x) < 0$ for $2 < x < 3$. Therefore:

$$H(x) \text{ is increasing on } (-3, 2) \quad \text{and decreasing on } (2, 3).$$

6. The graph of $f(x)$ is piecewise linear: a triangle from 0 to 2, a rectangle from 2 to 4, and another triangle from 4 to 6.

(a)

$$F(0) = \int_0^0 f(t)\,dt = 0,$$

$$F(2) = \int_0^2 f(t)\,dt = \tfrac{1}{2} \cdot 2 \cdot 2 = 2,$$

$$F(4) = \int_0^4 f(t)\,dt = \underbrace{\int_0^2 f(t)\,dt}_{2} + \underbrace{\int_2^4 f(t)\,dt}_{2\cdot 2=4} = 2 + 4 = 6,$$

$$F(6) = \int_0^6 f(t)\,dt = 2 + 4 + \underbrace{\tfrac{1}{2} \cdot 2 \cdot 2}_{2} = 8.$$

(b) By the Fundamental Theorem,

$$F'(x) = f(x).$$

(c) F is increasing where $F'(x) > 0$, i.e. where $f(x) > 0$. From the graph, $f(x) > 0$ for $0 < x < 6$ and $f(x) = 0$ at $x = 0$ and $x = 6$. Thus F is strictly increasing on $(0, 6)$ and never decreasing on $[0, 6]$ (there are no intervals where $f(x) < 0$).

(d) Because $F(x)$ represents the accumulated area under f from 0 to x, and $f(x) \geq 0$ on $[0, 6]$, $F(x)$ increases as x moves to the right. Therefore $F(x)$ is largest at the right endpoint,

$$F(6) = 8,$$

so the maximum of F on $[0, 6]$ occurs at $x = 6$.

7. Let

$$F(x) = \int_0^x (t^2 - 4t + 3)\, dt.$$

(a) By the Fundamental Theorem,

$$F'(x) = x^2 - 4x + 3.$$

Differentiate again:

$$F''(x) = 2x - 4.$$

(b) Critical numbers occur where $F'(x) = 0$:

$$x^2 - 4x + 3 = 0 \quad \Longrightarrow \quad (x-1)(x-3) = 0,$$

so $x = 1$ and $x = 3$ are critical numbers.

Use the second derivative test:

$$F''(1) = 2(1) - 4 = -2 < 0 \quad \Rightarrow \quad x = 1 \text{ is a local maximum.}$$

$$F''(3) = 2(3) - 4 = 2 > 0 \quad \Rightarrow \quad x = 3 \text{ is a local minimum.}$$

(c) Since

$$F'(x) = (x-1)(x-3),$$

its sign is:

$$F'(x) > 0 \text{ for } x < 1 \text{ and } x > 3, \qquad F'(x) < 0 \text{ for } 1 < x < 3.$$

Therefore:

$$F(x) \text{ is increasing on } (-\infty, 1) \text{ and } (3, \infty),$$

and

$$F(x) \text{ is decreasing on } (1, 3).$$

5.4 - FTC Part 2 and Indefinite Integrals

1.
$$\int (2e^x - 3x^2 + 4)\,dx = 2e^x - x^3 + 4x + C,$$
using linearity and the basic rules $\int e^x dx = e^x$, $\int x^2 dx = x^3/3$, $\int 1\,dx = x$.

2. (a)
$$\begin{aligned}\int \left(5x^4 - 7x^{-2} + 3x^{1/2}\right) dx &= 5 \cdot \frac{x^5}{5} - 7 \cdot \frac{x^{-1}}{-1} + 3 \cdot \frac{x^{3/2}}{3/2} + C \\ &= x^5 + 7x^{-1} + 2x^{3/2} + C.\end{aligned}$$

   (b)
$$\int \left(4\cos x - 7\sec^2 x\right) dx = 4\sin x - 7\tan x + C.$$

   (c) (Assume $x > 0$.)
$$\int \left(e^{3x} + \frac{1}{x}\right) dx = \frac{1}{3}e^{3x} + \ln x + C.$$

3. By the Fundamental Theorem of Calculus,
$$\int_{-1}^{3} k'(x)\,dx = k(3) - k(-1) = (-2) - 4 = -6.$$

4. Since F is an antiderivative of f, $\int_a^b f(x)\,dx = F(b) - F(a)$.

   (a)
$$\int_1^4 f(x)\,dx = F(4) - F(1) = 7 - 2 = 5.$$

   (b)
$$\int_4^1 f(x)\,dx = F(1) - F(4) = 2 - 7 = -5.$$

   (c)
$$\int_{-1}^4 f(x)\,dx = F(4) - F(-1) = 7 - 0 = 7.$$

5. (a)
$$\begin{aligned}\int_0^2 (x^2 - 4x + 5)\,dx &= \left[\frac{x^3}{3} - 2x^2 + 5x\right]_0^2 \\ &= \left(\frac{8}{3} - 8 + 10\right) - 0 = \frac{14}{3}.\end{aligned}$$

   (b)
$$\begin{aligned}\int_0^{\pi/2} (3\cos x - 2\sin x)\,dx &= [3\sin x + 2\cos x]_0^{\pi/2} \\ &= (3 \cdot 1 + 2 \cdot 0) - (3 \cdot 0 + 2 \cdot 1) = 3 - 2 = 1.\end{aligned}$$

(c)

$$\begin{aligned}\int_1^4 \left(\frac{2}{x} - \sqrt{x}\right) dx &= \left[2\ln x - \frac{2}{3}x^{3/2}\right]_1^4 \\ &= \left(2\ln 4 - \frac{2}{3}\cdot 4^{3/2}\right) - \left(2\ln 1 - \frac{2}{3}\cdot 1\right) \\ &= \left(2\ln 4 - \frac{16}{3}\right) - \left(0 - \frac{2}{3}\right) \\ &= 2\ln 4 - \frac{14}{3} = 4\ln 2 - \frac{14}{3}.\end{aligned}$$

6. We know $F'(x) = 3x^2 - 1$ and $F(0) = 5$.

(a)

$$F(x) = \int (3x^2 - 1)\, dx = x^3 - x + C.$$

Use $F(0) = 5$: $0 - 0 + C = 5 \Rightarrow C = 5$. Thus

$$F(x) = x^3 - x + 5.$$

(b)

$$\frac{d}{dx}(x^3 - x + 5) = 3x^2 - 1 = F'(x),$$

so the formula is correct.

(c)

$$F(2) = 2^3 - 2 + 5 = 11, \quad F(0) = 5, \quad F(2) - F(0) = 11 - 5 = 6.$$

Also,

$$\int_0^2 (3x^2 - 1)\, dx = \left[x^3 - x\right]_0^2 = (8 - 2) - 0 = 6.$$

Thus $F(2) - F(0) = \int_0^2 (3x^2 - 1)\, dx$, as FTC Part 2 predicts.

7. $P'(t) = 100 - 20t$ (thousand per year), $P(0) = 500$ (thousand).

(a)

$$\begin{aligned}\int_0^4 P'(t)\, dt &= \int_0^4 (100 - 20t)\, dt \\ &= \left[100t - 10t^2\right]_0^4 \\ &= (400 - 160) - 0 = 240.\end{aligned}$$

So the net change is 240 thousand organisms.

(b)

$$P(4) = P(0) + \text{net change} = 500 + 240 = 740$$

(thousand organisms).

(c) This illustrates FTC Part 2 because the integral of the rate of change equals the net change:

$$P(4) - P(0) = \int_0^4 P'(t)\,dt.$$

5.5 - The Substitution Rule

1. Let $u = x^2 + 1$. Then $du = 2x\,dx$, so $3x\,dx = \frac{3}{2}\,du$.

$$\begin{aligned}\int \frac{3x}{x^2+1}\,dx &= \int \frac{3x}{u}\,dx = \int \frac{3}{2}\frac{1}{u}\,du \\ &= \frac{3}{2}\ln|u| + C = \frac{3}{2}\ln(x^2+1) + C.\end{aligned}$$

$$\boxed{\int \frac{3x}{x^2+1}\,dx = \frac{3}{2}\ln(x^2+1) + C}$$

2. Let $u = x^2 + 9$. Then $du = 2x\,dx$, so $4x\,dx = 2\,du$.

$$\begin{aligned}\int 4x\sqrt{x^2+9}\,dx &= \int 4x\,\sqrt{u}\,dx = \int 2u^{1/2}\,du \\ &= 2\cdot\frac{2}{3}u^{3/2} + C = \frac{4}{3}(x^2+9)^{3/2} + C.\end{aligned}$$

$$\boxed{\int 4x\sqrt{x^2+9}\,dx = \frac{4}{3}(x^2+9)^{3/2} + C}$$

3. Let $u = x^2 - x$. Then $du = (2x-1)\,dx$.

$$\begin{aligned}\int (2x-1)\,e^{x^2-x}\,dx &= \int e^u\,du \\ &= e^u + C = e^{x^2-x} + C.\end{aligned}$$

$$\boxed{\int (2x-1)\,e^{x^2-x}\,dx = e^{x^2-x} + C}$$

4. Let $u = \sin(3x)$. Then $du = 3\cos(3x)\,dx$, so $\cos(3x)\,dx = \frac{1}{3}\,du$.

$$\begin{aligned}\int \cos(3x)\,\sin(3x)\,dx &= \int u\cdot\cos(3x)\,dx = \int u\cdot\frac{1}{3}\,du \\ &= \frac{1}{3}\int u\,du = \frac{1}{3}\cdot\frac{u^2}{2} + C = \frac{1}{6}\sin^2(3x) + C.\end{aligned}$$

$$\boxed{\int \cos(3x)\,\sin(3x)\,dx = \frac{1}{6}\sin^2(3x) + C}$$

5. For $x > 1$, let $u = \ln x$. Then $du = \frac{1}{x}\,dx$, so $\frac{1}{x\ln x}\,dx = \frac{1}{u}\,du$.

$$\begin{aligned}\int \frac{1}{x\ln x}\,dx &= \int \frac{1}{u}\,du = \ln|u| + C \\ &= \ln(\ln x) + C \quad (\text{since } x > 1 \implies \ln x > 0).\end{aligned}$$

$$\boxed{\int \frac{1}{x\ln x}\,dx = \ln(\ln x) + C}$$

6. Let $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}du$. Change the limits: when $x = 0$, $u = 1$; when $x = 2$, $u = 5$.

$$\begin{aligned} \int_0^2 \frac{x}{x^2+1}\,dx &= \int_1^5 \frac{1}{2}\,\frac{1}{u}\,du \\ &= \frac{1}{2}\int_1^5 \frac{1}{u}\,du = \frac{1}{2}\big[\ln|u|\big]_1^5 \\ &= \frac{1}{2}\big(\ln 5 - \ln 1\big) = \frac{1}{2}\ln 5. \end{aligned}$$

$$\boxed{\int_0^2 \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln 5}$$

7. Let $u = \cos x$. Then $du = -\sin x\,dx$, so $\sin x\,dx = -du$. When $x = 0$, $u = \cos 0 = 1$; when $x = \frac{\pi}{4}$, $u = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \cos^2 x\,\sin x\,dx &= \int_1^{\frac{\sqrt{2}}{2}} u^2(-du) \\ &= -\int_1^{\frac{\sqrt{2}}{2}} u^2\,du = \int_{\frac{\sqrt{2}}{2}}^1 u^2\,du \\ &= \left[\frac{u^3}{3}\right]_{\frac{\sqrt{2}}{2}}^1 = \frac{1}{3} - \frac{\left(\frac{\sqrt{2}}{2}\right)^3}{3} \\ &= \frac{1}{3} - \frac{\sqrt{2}}{12} = \frac{4-\sqrt{2}}{12}. \end{aligned}$$

$$\boxed{\int_0^{\frac{\pi}{4}} \cos^2 x\,\sin x\,dx = \frac{4-\sqrt{2}}{12}}$$

8. Let $u = x^2 + 3$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$. When $x = 1$, $u = 4$; when $x = 4$, $u = 19$.

$$\begin{aligned} \int_1^4 \frac{x}{\sqrt{x^2+3}}\,dx &= \int_4^{19} \frac{1}{2}\,u^{-1/2}\,du \\ &= \frac{1}{2}\cdot\frac{2}{1}u^{1/2}\Big|_4^{19} = \sqrt{u}\Big|_4^{19} \\ &= \sqrt{19} - \sqrt{4} = \sqrt{19} - 2. \end{aligned}$$

$$\boxed{\int_1^4 \frac{x}{\sqrt{x^2+3}}\,dx = \sqrt{19} - 2}$$

9. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$. When $x = 0$, $u = 1$; when $x = 1$, $u = 2$.

$$\begin{aligned}
\int_0^1 x\sqrt{1+x^2}\,dx &= \int_1^2 \frac{1}{2}u^{1/2}\,du \\
&= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^2 = \frac{1}{3}\Big(u^{3/2}\Big|_1^2\Big) \\
&= \frac{1}{3}\big(2^{3/2} - 1\big) = \frac{1}{3}\big(2\sqrt{2} - 1\big).
\end{aligned}$$

$$\boxed{\int_0^1 x\sqrt{1+x^2}\,dx = \frac{1}{3}\big(2\sqrt{2} - 1\big)}$$

6.1 - Areas Between Curves

1. **Region between $y = 4 - x^2$ and $y = -2x + 4$.**

   Intersection points:

$$\begin{aligned} 4 - x^2 &= -2x + 4 \\ -x^2 + 2x &= 0 \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \quad \Rightarrow \quad x = 0,\, 2. \end{aligned}$$

   On $[0, 2]$,

$$y_{\text{top}} = 4 - x^2, \qquad y_{\text{bottom}} = -2x + 4.$$

   Area:

$$\begin{aligned} A &= \int_0^2 \big[y_{\text{top}} - y_{\text{bottom}}\big]\, dx \\ &= \int_0^2 \big[(4 - x^2) - (-2x + 4)\big]\, dx \\ &= \int_0^2 (-x^2 + 2x)\, dx \\ &= \left[-\frac{x^3}{3} + x^2\right]_0^2 = \left(-\frac{8}{3} + 4\right) - 0 = \frac{4}{3}. \end{aligned}$$

2. **Region enclosed by $y = x^2$ and $y = 2x + 3$.**

   Intersection points:

$$\begin{aligned} x^2 &= 2x + 3 \\ x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x &= -1,\ 3. \end{aligned}$$

   On $[-1, 3]$, $y = 2x + 3$ is on top, $y = x^2$ is on bottom (e.g. at $x = 0$, $3 > 0$). Thus

$$\begin{aligned} A &= \int_{-1}^{3} \big[(2x+3) - x^2\big]\, dx \\ &= \int_{-1}^{3} (-x^2 + 2x + 3)\, dx \\ &= \left(-\frac{x^3}{3} + x^2 + 3x\right)_{-1}^{3} \\ &= \frac{32}{3}. \end{aligned}$$

3. **Region bounded by $y = x^2 + 1$ and $y = 3x - 1$.**

   Intersection points:

$$\begin{aligned} x^2 + 1 &= 3x - 1 \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \\ x &= 1,\ 2. \end{aligned}$$

   On $[1, 2]$, $y = 3x - 1$ is on top and $y = x^2 + 1$ is on bottom:

$$(3x - 1) - (x^2 + 1) = -x^2 + 3x - 2 > 0 \quad \text{for } 1 < x < 2.$$

   Hence

$$\begin{aligned} A &= \int_1^2 \big[(3x-1) - (x^2+1)\big]\,dx \\ &= \int_1^2 (-x^2 + 3x - 2)\,dx \\ &= \left(-\frac{x^3}{3} + \frac{3x^2}{2} - 2x\right)_1^2 \\ &= \frac{1}{6}. \end{aligned}$$

4. **Region bounded by $x = y^2$ and $x = 4 - y^2$.**

   (a) For each fixed y, the region runs from $x = y^2$ (left) to $x = 4 - y^2$ (right). A horizontal slice intersects each curve exactly once, so it is most natural to integrate with respect to y.

   (b) Intersections come from

$$y^2 = 4 - y^2 \quad \Rightarrow \quad 2y^2 = 4 \quad \Rightarrow \quad y = \pm\sqrt{2}.$$

   Thus

$$\begin{aligned} A &= \int_{-\sqrt{2}}^{\sqrt{2}} \big[(4 - y^2) - (y^2)\big]\,dy \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} (4 - 2y^2)\,dy \\ &= \left(4y - \frac{2y^3}{3}\right)_{-\sqrt{2}}^{\sqrt{2}} \\ &= \frac{16\sqrt{2}}{3}. \end{aligned}$$

5. **Region R bounded by $y = x^2$, $y = 4$, and $x = 0$.**

   (a) Vertical slices (with respect to x).

   The parabola $y = x^2$ meets $y = 4$ at $x^2 = 4$, so $x = \pm 2$. Because $x = 0$ is also a boundary, the region is

   $$0 \le x \le 2, \quad x^2 \le y \le 4.$$

   Thus

   $$A = \int_0^2 \left(4 - x^2\right) dx.$$

   (b) Horizontal slices (with respect to y).

   Solve $y = x^2$ for x on the right branch: $x = \sqrt{y}$. The left boundary is $x = 0$, and y ranges from 0 (at $(0, 0)$) up to 4 (at $(0, 4)$ and $(2, 4)$). So

   $$A = \int_0^4 \left(\sqrt{y} - 0\right) dy.$$

   (c) Evaluate the area.

   Using either integral:

   $$A = \int_0^2 (4 - x^2)\, dx = \left(4x - \frac{x^3}{3}\right)_0^2 = 8 - \frac{8}{3} = \frac{16}{3},$$

   or

   $$A = \int_0^4 \sqrt{y}\, dy = \left(\frac{2}{3} y^{3/2}\right)_0^4 = \frac{2}{3} \cdot 8 = \frac{16}{3}.$$

6. **Region enclosed by $y = |x|$ and $y = 2$.**

   (a) Intersections.

   Solve $|x| = 2$:

   $$|x| = 2 \quad \Rightarrow \quad x = -2,\ 2.$$

   (b) Integral(s) for the area.

   On $[-2, 2]$, the top curve is $y = 2$ and the bottom curve is $y = |x|$. A convenient expression using symmetry is

   $$A = \int_{-2}^2 \left(2 - |x|\right) dx = 2 \int_0^2 (2 - x)\, dx.$$

   (c) Evaluate.

   $$\begin{aligned} A &= 2 \int_0^2 (2 - x)\, dx = 2 \left(2x - \frac{x^2}{2}\right)_0^2 \\ &= 2\big[(4 - 2) - 0\big] = 2 \cdot 2 = 4. \end{aligned}$$