# Midterm 3 Study Guide

## MATH1300 - Calculus I

Fall 2025

## Contents


**3.6 - Derivatives of Inverse Trigonometric Functions** 1

**3.6 - Derivatives of Logarithmic Functions** 2

**3.7 - Rates of Change in the Natural and Social Sciences** 3

**3.10 - Linear Approximation** 4

**4.1 - Maximum and Minimum Values** 5

**4.2 - The Mean Value Theorem** 6

**4.3 & 4.5 - Curve Sketching** 7

**4.4 - Indeterminate Forms and l'Hôpital's Rule** 9

**4.7 - Applied Optimization** 10

**4.9 - Antiderivatives** 12

## 3.6 - Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\left(\csc^{-1}x\right) = -\frac{1}{|x|\sqrt{x^2-1}},$$

$$\frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{|x|\sqrt{x^2-1}},$$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}.$$

**Remark:** You will not be tested on $\csc^{-1}(x), \sec^{-1}(x)$, or $\cot^{-1}(x)$.

1. Let $g(x) = \sin^{-1}\big(3x^2 - 1\big)$. Compute $g'(x)$.

2. Let $h(u) = \cos^{-1}\big(\sqrt{u}\big)$. Compute $h'(u)$.

3. Let $p(z) = \tan^{-1}\big(1 + z^2\big)$. Compute $p'(z)$.

4. Let $y(x) = x\,\sin^{-1}\big(x^3\big)$. Compute $y'(x)$.

5. Let $v(s) = \cos^{-1}(2s - 5) + \tan^{-1}\big(s^2\big)$. Compute $v'(s)$.

6. Let $m(\theta) = \arcsin\big(\arctan\theta\big)$. Compute $m'(\theta)$.

7. Let $q(x) = \arccos\big(e^{-x}\big)$. Compute $q'(x)$.

8. Let $r(t) = \tan^{-1}\big(2t - 1\big)$. Compute $r'(t)$.

9. Let $w(r) = \dfrac{1}{\cos^{-1}(r)}$. Compute $w'(r)$.

10. Let $s(x) = \tan^{-1}\big((2x - 1)^5\big)$. Compute $s'(x)$.

## 3.6 - Derivatives of Logarithmic Functions

**Derivatives of Logarithmic Functions:**

- $\frac{d}{dx}\big(\log_b(x)\big) = \frac{1}{x\ln(b)}$
- $\frac{d}{dx}\big(\ln(x)\big) = \frac{1}{x}$

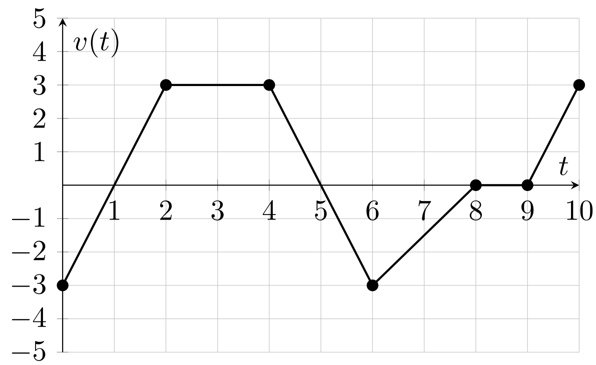
**Logarithmic Differentiation:** Logarithmic differentiation is a technique for differentiating functions where $x$ appears in both the base and exponent (e.g., $f(x)^{g(x)}$), or when products/quotients/powers are messy.

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to expand the expression.
2. Differentiate implicitly with respect to $x$.
3. Solve the resulting equation for $y'$ and replace $y$ by $f(x)$.

1. Compute $\frac{d}{dx}\log_2\big(\sqrt{1+x^3}\big)$.
2. Compute $\frac{d}{dx}\big[x\log_{10}(2x+3)\big]$.
3. Compute $\frac{d}{dx}\big(\ln(5-2x)\big)^3$.
4. Compute $\frac{d}{dx}\frac{\ln(x^2)}{1+x}$.
5. Compute $\frac{d}{dx}\ln\left(\frac{x^2+1}{\sqrt{1-x}}\right)$.
6. Let $f(x) = (x^2+1)^{\sqrt{x}}$. Find $f'(x)$.
7. Let $f(x) = \left(\frac{1+x}{1-x}\right)^{x^2}$. Find $f'(x)$.
8. Let $f(x) = (2x-1)^x$. Find $f'(x)$.
9. Let $f(x) = x^{\sin x}$ (assume $x > 0$). Find $f'(x)$.
10. Let $f(x) = (\ln x)^x$ (assume $x > 1$). Find $f'(x)$.

## 3.7 - Rates of Change in the Natural and Social Sciences

1. Suppose that the graph of the velocity function of a particle is as shown in the figure, where $t$ is measured in seconds.




   (a) From the graph above, on which time intervals is the particle *moving forward*?

   (b) List all times when the particle is *at rest.* Explain in terms of the graph of $v(t)$.

   (c) On which intervals is the particle *speeding up*?

2. If a ball is thrown straight up with initial velocity 96 ft/s, its height after $t$ seconds is

$$s(t) = 96t - 16t^2.$$

   (a) What is the velocity of the ball after 1 second?

   (b) What is the maximum height reached by the ball?

   (c) What is the velocity of the ball when it is 128 ft above the ground on its way up?

3. A culture of yeast cells quadruples every hour and begins with 80 cells.

   (a) Find a formula for the number $P(t)$ of cells after $t$ hours.

   (b) Use it to estimate the rate of growth at $t = 0.75$ hours.

4. The cost to produce $q$ gadgets is $C(q) = 5400 + 100\sqrt{q}$.

   (i) Find $C'(q)$ and evaluate at $q = 400$.

   (ii) Explain the meaning of your result and give an estimate for the cost of the 401$^{\text{st}}$ gadget.

   (iii) Compute the actual cost of the 401$^{\text{st}}$ gadget.

## 3.10 - Linear Approximation

For a differentiable function $f$ at $x = a$, the best local linear model for $f$ near $a$ is its tangent line:

$$L(x) = f(a) + f'(a)\,(x - a).$$

Use $L$ to estimate $f(x_0)$ when $x_0$ is close to $a$: compute $f(a)$, $f'(a)$, write $L(x)$, then plug in $x_0$.

- If $f''(x) > 0$ near $a$ (concave up), then $L(x)$ *underestimates* $f(x)$.
- If $f''(x) < 0$ near $a$ (concave down), then $L(x)$ *overestimates* $f(x)$.

1. Let $f(x) = \cos x$ at $a = \frac{\pi}{3}$. Find the linearization $L(x)$.

2. Let $g(x) = \sqrt{4 - x}$ at $a = 1$. Find the linearization $L(x)$.

3. Let $h(x) = e^{2x}$ at $a = 0$. Find the linearization $L(x)$.

4. Let $f(x) = \sqrt[3]{x}$ and linearize at $a = 8$.
   - Use your linearization to estimate $\sqrt[3]{7.9}$.
   - Is this estimate an overestimate or an underestimate? Explain.

5. Let $g(x) = \sin x$ and linearize at $a = 0$.
   - Use your linearization to estimate $\sin(0.76)$.
   - Is the estimate an overestimate or an underestimate? Explain.

6. Let $h(x) = \ln x$ and linearize at $a = 2$.
   - Use your linearization to estimate $\ln(2.05)$.
   - Is the estimate an overestimate or an underestimate? Explain.

## 4.1 - Maximum and Minimum Values

A **critical number** of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

**The Closed Interval Method.** To find the *absolute* maximum and minimum values of a continuous function $f$ on a closed interval $[a, b]$:

1. Find the critical numbers.
2. Find the values of $f$ at the critical numbers of $f$ in $(a, b)$.
3. Find the values of $f$ at the endpoints of the interval.
4. The largest of the values from Steps 1-3 is the absolute maximum value; the smallest of these values is the absolute minimum value.

1. Find all critical numbers of $f(x) = x^{\frac{2}{3}}(x - 3)$.
2. Find all critical numbers of $f(x) = \dfrac{x^2 + 4x + 5}{x + 1}$.
3. Find all critical numbers of $f(x) = \dfrac{x - 1}{\sqrt{5 - x}}$.
4. Find all critical numbers of $f(x) = x\,e^{-x^2}$.
5. Find the absolute maximum and minimum value of $f(x) = x^3 - 3x$ on $[-2, 2]$.
6. Find the absolute maximum and minimum value of $f(x) = \sqrt{9 - x^2}$ on $[-3, 2]$.
7. Find the absolute maximum and minimum value of $f(x) = \ln x - \dfrac{x}{2}$ on $[1, 6]$.
8. Find the absolute maximum and minimum value of $f(x) = (x - 1)^{2/3}$ on $[-2, 4]$.

## 4.2 - The Mean Value Theorem

**Rolle's Theorem** Let $f$ be a function that satisfies the following:

1. $f$ is continuous on the closed interval $[a, b]$.
2. $f$ is differentiable on the open interval $(a, b)$.
3. $f(a) = f(b)$.

Then there is a number $c$ in $(a, b)$ such that $f'(c) = 0$.

**Mean Value Theorem** Let $f$ be a function that satisfies the following:

1. $f$ is continuous on the closed interval $[a, b]$.
2. $f$ is differentiable on the open interval $(a, b)$.

Then there exists a number $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Equivalently,

$$f(b) - f(a) = f'(c)\,(b - a).$$

1. Suppose $f$ is continuous on $[2, 7]$ and differentiable on $(2, 7)$ with $f(2) = 5$ and $f(7) = -1$. State a conclusion guaranteed by the Mean Value Theorem and justify it.
2. Let $f$ be continuous on $[0, 4]$ and differentiable on $(0, 4)$ with $f(0) = 3$ and $f(4) = 3$. What does the Mean Value Theorem guarantee? Justify.
3. A car's position $s(t)$ (in miles) is continuous on $[0, 1.5]$ and differentiable on $(0, 1.5)$. If $s(0) = 0$ and $s(1.5) = 120$, state the speed guaranteed by the Mean Value Theorem and when it occurs. Justify.
4. Suppose $g$ is continuous on $[0, 6]$ and differentiable on $(0, 6)$ with $g(0) = g(3) = g(6)$. Show that there exist *at least two* numbers $c_1, c_2$ in $(0, 6)$ with $g'(c_1) = g'(c_2) = 0$. Name the theorem(s) used and justify.
5. Let $f$ be continuous on $[0, 2]$ and differentiable on $(0, 2)$ with $f(0) = -1$ and $f(2) = 7$. Draw the secant line through the endpoints and state the Mean Value Theorem conclusion about a tangent line in $(0, 2)$. Name the slope that must occur and justify.

## 4.3 & 4.5 - Curve Sketching

To build a clear picture of a function, we can understand its behavior from its derivatives. The first derivative locates and *classifies* critical numbers and tells where the function is increasing or decreasing; the second derivative reveals *concavity* and pinpoints possible inflection points. Listed below are steps to follow in order to analyze–and eventually graph–a function $f(x)$.

- **Domain of $f(x)$.** Find the domain $D$ of $f$. Note any points in $D$ where $f'$ or $f''$ does not exist (corners, cusps, vertical tangents).
- **Critical numbers.** Compute $f'(x)$. A *critical number* is any $c$ in the domain of $f(x)$ where

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

  List all such $c$.

- **Classify each critical number.**
  1. Sort the critical numbers on a number line; pick one test value in each interval.
  2. Make a sign chart for $f'(x)$. Then apply the First Derivative Test:

$$\begin{aligned} f' : + \to - \quad &\Rightarrow \quad \text{local maximum at } c, \\ f' : - \to + \quad &\Rightarrow \quad \text{local minimum at } c, \\ f' : \text{no sign change} \quad &\Rightarrow \quad \text{neither (saddle/flat).} \end{aligned}$$

  3. (Optional) Second derivative test at $c$: if $f'(c) = 0$ and $f''(c) > 0$ then local min; if $f''(c) < 0$ then local max; if $f''(c) = 0$, test is inconclusive—use the first derivative test.

- **Intervals of increase/decrease.** From the $f'$ sign chart:

$$f'(x) > 0 \;\Rightarrow\; f \text{ increasing,} \qquad f'(x) < 0 \;\Rightarrow\; f \text{ decreasing.}$$

- **Concavity and inflection points.**
  1. Compute $f''(x)$. Candidates for concavity changes are solutions of $f''(x) = 0$ and points in $D$ where $f''$ does not exist.
  2. Make a sign chart for $f''(x)$. Then:

$$f''(x) > 0 \;\Rightarrow\; \text{concave up,} \qquad f''(x) < 0 \;\Rightarrow\; \text{concave down.}$$

  3. *Inflection points:* $x = c$ is an inflection point if $c$ is in the domain $D$, $f$ is continuous at $c$, and the sign of $f''$ changes across $c$.

**For Problems 1–5, find:** (a) The critical numbers; (b) Local extrema; (c) Increasing/Decreasing intervals; (d) Inflection points; (e) Concavity intervals.

1. $f(x) = x^3 - 6x^2 + 9x$
2. $f(x) = \dfrac{x}{x^2 + 4}$
3. $f(x) = (x + 1)e^{-2x}$
4. $f(x) = x^5 - 10x^3$
5. $f(x) = x^4(x - 2)$
6. Give a differentiable function with $f'(0) = f''(0) = 0$ that has:

   (i) a local *maximum* at 0;

   (ii) an *inflection point* at 0 but no local maximum or minimum.

7. Suppose

   $$f''(x) = (x + 3)^5(x - 1)^4(x - 5)^3.$$

   Find all inflection $x$-values and the open intervals of concavity. Briefly explain how *factor multiplicity* (odd vs. even) controls sign changes.

8. Suppose

   $$f''(x) = (x - 2)^2(x + 1)^3(x - 4).$$

   Find the inflection $x$-values and the open intervals of concavity.

9. Suppose

   $$f''(x) = (x + 2)(x - 1)^2(x - 3).$$

   Find the $x$-values of all inflection points and the open intervals of concavity.

*On separate axes, sketch a continuous $f$ satisfying each set of conditions.*

10. Pass through $(-3, 1)$ and $(2, -1)$; critical numbers only at $x = -1, 1, 4$; concave up on $(-\infty, -1)$ and $(2, 4)$; concave down on $(-1, 2)$ and $(4, \infty)$; global *minimum* at $x = -1$.
11. Pass through $(0, 0)$ and $(4, 2)$; critical numbers only at $x = 0, 2, 5$; $f''(x) > 0$ on $(0, 1)$ and $(3, 5)$; $f''(x) < 0$ on $(-\infty, 0)$, $(1, 3)$, and $(5, \infty)$; global *maximum* at $x = 5$.
12. Pass through $(-2, -1)$ and $(3, 1)$; critical numbers only at $x = -2, 1, 3$; concave up on $(-\infty, -2)$ and $(1, 3)$; concave down on $(-2, 1)$ and $(3, \infty)$; local maxima at $x = -2$ and $x = 3$ with the latter higher.

# 4.4 - Indeterminate Forms and l'Hôpital's Rule

**Indeterminate Differences**

$\infty - \infty, \quad -\infty - (-\infty)$

Convert to a quotient using these options:

- combine using a common denominator
- rationalize (conjugate) when roots are present
- factor out the dominant term and compare

**Indeterminate Products**

$0\cdot\infty,\ 0\cdot(-\infty),\ \infty\cdot 0,\ (-\infty)\cdot 0$

Convert to a quotient:

$f\cdot g = \dfrac{f}{1/g}$ or $f\cdot g = \dfrac{g}{1/f}$.

**Indeterminate Powers**

$0^0,\ \infty^0,\ 1^\infty$

Convert via logs: Let $y = f^{\,g}$. Then $\ln y = g\ln f$ (a product). Rewrite as a *quotient*; compute the limit, then exponentiate to get $y$.

**Quotient forms eligible for l'Hôpital:**

$\dfrac{0}{0},\quad \dfrac{\infty}{\infty},\quad \dfrac{-\infty}{\infty},\quad \dfrac{\infty}{-\infty},\quad \dfrac{-\infty}{-\infty}$

**Apply l'Hôpital's Rule:**

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

1. $\displaystyle\lim_{x\to 0}\frac{e^{2x}-1-2x}{x^2}$
2. $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)-x+\frac{1}{2}x^2}{x^3}$
3. $\displaystyle\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$
4. $\displaystyle\lim_{x\to\infty}\frac{2x^2+3x+1}{x^2-5x+4}$
5. $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$
6. $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+4x}-x\right)$
7. $\displaystyle\lim_{x\to 0^+} x\ln x$
8. $\displaystyle\lim_{x\to 0^+} x\cot x$
9. $\displaystyle\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$
10. $\displaystyle\lim_{x\to 0^+}\left(\sin x\right)^{x}$

## 4.7 - Applied Optimization

**Optimization Workflow**

1. Define variables and write the *objective* $f$ in one variable using constraints.
2. Identify the domain (including any endpoints) that makes sense for the problem.
3. Compute $f'(x)$ and solve $f'(x) = 0$ to find interior critical numbers.
4. Use the tools below to justify where the extremum occurs and answer the question.

**Closed Interval Method (Absolute Extrema):** If $f$ continuous on $[a, b]$:

1. Find interior critical numbers in $(a, b)$.
2. Evaluate $f$ at those points and at $a, b$.
3. Largest value $\Rightarrow$ absolute max; smallest value $\Rightarrow$ absolute min.

**First Derivative Test (Local Extrema):** Let $c$ be a critical number of a continuous $f$. Make a sign chart for $f'$ near $c$:

$$\begin{aligned} f' : + \to - &\quad\Rightarrow\quad \text{local max at } c,\\ f' : - \to + &\quad\Rightarrow\quad \text{local min at } c,\\ \text{no sign change} &\quad\Rightarrow\quad \text{neither.} \end{aligned}$$

*Monotonicity:* $f' > 0 \Rightarrow$ increasing, $f' < 0 \Rightarrow$ decreasing.

**Second Derivative Test (Local Extrema):** If $f''$ is continuous near $c$ and $f'(c) = 0$.

$$f''(c) > 0 \Rightarrow \text{local min}, \qquad f''(c) < 0 \Rightarrow \text{local max}, \qquad f''(c) = 0 \Rightarrow \text{inconclusive.}$$

(If $f'$ or $f''$ does not exist at $c$, this test cannot be applied.)

**First Derivative Test for *Absolute* Extremes:** If $f$ is continuous on an interval and $c$ is a critical number, then:

$$\begin{aligned} f'(x) > 0 \text{ for } x < c \text{ and } f'(x) < 0 \text{ for } x > c &\quad\Rightarrow\quad f(c) \text{ is an absolute maximum},\\ f'(x) < 0 \text{ for } x < c \text{ and } f'(x) > 0 \text{ for } x > c &\quad\Rightarrow\quad f(c) \text{ is an absolute minimum}. \end{aligned}$$

(Use when the sign of $f'$ is known on the whole interval.)

1. A rectangle has perimeter 120 m. Find the dimensions that maximize its area.
2. An open-top box is made by cutting congruent squares of side $x$ from the corners of a 20 cm $\times$ 30 cm sheet and folding up the sides. For what $x$ is the volume maximized?
3. A right circular cylinder must hold $500\,\text{cm}^3$ of liquid. Find the radius and height that minimize the surface area.
4. A rectangle is inscribed in the region under $y = 9 - x^2$ and above the $x$-axis. Its base lies on the $x$-axis and its upper vertices touch the parabola. Find the dimensions that maximize the rectangle's area.
5. A farmer has 100 m of fencing to enclose a rectangular pen against a straight river (no fence along the river). What dimensions maximize the area?
6. Two positive numbers have product 64. Find the numbers that minimize their sum.
7. A shop sells a gadget for \$60 and sells 40 units per week. For each \$3 decrease in price, weekly sales increase by 8 units. What price maximizes weekly *revenue*? Justify.
8. A poster must contain $400\,\text{cm}^2$ of printed text. It has margins of 3 cm at the top and bottom and 2 cm on each side. Find the overall dimensions that minimize the total paper area.
9. A lifeguard runs along a straight beach at 6 m/s and swims at 2 m/s. A swimmer is 50 m offshore at a point 80 m down the beach from the lifeguard's current position. Where should the lifeguard enter the water to minimize the rescue time?
10. A 100 cm piece of wire is cut into two pieces: one is bent into a circle, the other into a square. How should the wire be cut to *minimize* the total area enclosed? (Also: how to *maximize* it?)

## 4.9 - Antiderivatives

An antiderivative of $f$ is any function $F$ with $F'(x) = f(x)$. All antiderivatives differ by an additive constant $C$. A specific antiderivative can be found by giving an initial condition $F(a) = b$.

| $f(x)$ | General Antiderivative $F(x)$ |
|---|---|
| $k$ (constant) | $kx + C$ |
| $x^n \ (n \neq -1)$ | $\dfrac{x^{n+1}}{n+1} + C$ |
| $\dfrac{1}{x} \ (x \neq 0)$ | $\ln\lvert x\rvert + C$ |
| $e^{ax}$ | $\dfrac{1}{a}e^{ax} + C$ |
| $a^x \ (a > 0,\ a \neq 1)$ | $\dfrac{a^x}{\ln a} + C$ |
| $\sin x$ | $-\cos x + C$ |
| $\cos x$ | $\sin x + C$ |
| $\sec^2 x$ | $\tan x + C$ |
| $\sec x \tan x$ | $\sec x + C$ |
| $\dfrac{1}{1+x^2}$ | $\arctan x + C$ |
| $x^{1/2}$ | $\dfrac{2}{3}x^{3/2} + C$ |
| $x^{-3/2}$ | $-2x^{-1/2} + C$ |

1. Find the most general antiderivative of $f(x) = 3x^2 - 4x + \cos x$.
2. Find the most general antiderivative of $f(x) = e^{2x} - 5$.
3. Find the most general antiderivative of $f(x) = \sec^2 x - 2x$.
4. Find the most general antiderivative of $f(x) = x^{-3/2} + 5x^{1/2}$.
5. Find the most general antiderivative of $f(x) = \dfrac{1}{x} + \dfrac{1}{1+x^2}$.
6. Find the most general antiderivative of $f(x) = \cos(3x) - 4\sin(2x)$.
7. Let $F'(x) = 2x\,e^{x^2}$ and $F(0) = 5$. Find $F(x)$.
8. Let $F'(x) = \sqrt{x} + \dfrac{1}{x}$ and $F(1) = 0$. Find $F(x)$.
9. Let $F'(x) = \dfrac{1}{\sqrt{1-x^2}}$ on $(-1, 1)$ and $F(0) = 0$. Find $F(x)$.
10. Let $F'(x) = e^{-x} - 3x^2 + 2$ and $F(2) = 7$. Find $F(x)$.