Midterm 3 Study Guide – Solutions

MATH1300 - Calculus I

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3.6 - Derivatives of Inverse Trigonometric Functions

1. $g'(x) = \dfrac{\mathrm{d}}{\mathrm{d}x} \arcsin\big(3x^2 - 1\big) = \dfrac{6x}{\sqrt{1 - \big(3x^2-1\big)^2}}.$

2. $h'(u) = \dfrac{\mathrm{d}}{\mathrm{d}u} \arccos\big(\sqrt{u}\,\big) = -\dfrac{\frac{1}{2\sqrt{u}}}{\sqrt{1-(\sqrt{u})^2}} = -\dfrac{1}{2\sqrt{u}\,\sqrt{1-u}}.$

3. $p'(z) = \dfrac{\mathrm{d}}{\mathrm{d}z} \arctan\big(1+z^2\big) = \dfrac{2z}{1+\big(1+z^2\big)^2}.$

4. $y'(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\Big(x \arcsin(x^3)\Big) = \arcsin(x^3) + \dfrac{3x^3}{\sqrt{1-x^6}}.$

5. $v'(s) = \dfrac{\mathrm{d}}{\mathrm{d}s} \arccos(2s-5) + \dfrac{\mathrm{d}}{\mathrm{d}s} \arctan(s^2) = -\dfrac{2}{\sqrt{1-(2s-5)^2}} + \dfrac{2s}{1+s^4}.$

6. $m'(\theta) = \dfrac{\mathrm{d}}{\mathrm{d}\theta} \arcsin(\arctan\theta) = \dfrac{\frac{1}{1+\theta^2}}{\sqrt{1-(\arctan\theta)^2}} = \dfrac{1}{(1+\theta^2)\sqrt{1-(\arctan\theta)^2}}.$

7. $q'(x) = \dfrac{\mathrm{d}}{\mathrm{d}x} \arccos(e^{-x}) = -\dfrac{-e^{-x}}{\sqrt{1-e^{-2x}}} = \dfrac{e^{-x}}{\sqrt{1-e^{-2x}}}.$

8. $r'(t) = \dfrac{\mathrm{d}}{\mathrm{d}t} \arctan(2t-1) = \dfrac{2}{1+(2t-1)^2}.$

9. $w'(r) = \dfrac{\mathrm{d}}{\mathrm{d}r}\,\big[\arccos(r)\big]^{-1} = -\big[\arccos(r)\big]^{-2} \cdot \Big(-\dfrac{1}{\sqrt{1-r^2}}\Big) = \dfrac{1}{\arccos(r)^2\sqrt{1-r^2}}.$

10. $s'(x) = \dfrac{\mathrm{d}}{\mathrm{d}x} \arctan\big((2x-1)^5\big) = \dfrac{10(2x-1)^4}{1+(2x-1)^{10}}.$

3.6 - Derivatives of Logarithmic Functions

1. $\dfrac{d}{dx}\,\log_2\!\left(\sqrt{1+x^3}\right) = \dfrac{3x^2}{2(1+x^3)\,\ln 2}$

2. $\dfrac{d}{dx}\,\left[x\,\log_{10}(2x+3)\right] = \log_{10}(2x+3) + \dfrac{2x}{(2x+3)\,\ln 10}$

3. $\dfrac{d}{dx}\,\left(\ln(5-2x)\right)^3 = -\dfrac{6\left(\ln(5-2x)\right)^2}{5-2x}$

4. $\dfrac{d}{dx}\,\dfrac{\ln(x^2)}{1+x} = \dfrac{\frac{2}{x}(1+x) - \ln(x^2)}{(1+x)^2}$

5. $\dfrac{d}{dx}\,\ln\!\left(\dfrac{x^2+1}{\sqrt{1-x}}\right) = \dfrac{2x}{x^2+1} + \dfrac{1}{2(1-x)}$

6. $f'(x) = (x^2+1)^{\sqrt{x}}\left(\dfrac{\ln(x^2+1)}{2\sqrt{x}} + \dfrac{2x\sqrt{x}}{x^2+1}\right)$

7. $f'(x) = \left(\dfrac{1+x}{1-x}\right)^{x^2}\left(2x\left[\ln(1+x) - \ln(1-x)\right] + x^2\left[\dfrac{1}{1+x} + \dfrac{1}{1-x}\right]\right)$

8. $f'(x) = (2x-1)^x\left(\ln(2x-1) + \dfrac{2x}{2x-1}\right)$

9. $f'(x) = x^{\sin x}\left(\cos x\,\ln x + \dfrac{\sin x}{x}\right)$

10. $f'(x) = (\ln x)^x\left(\ln\ln x + \dfrac{1}{\ln x}\right)$

3.7 - Rates of Change in the Natural and Social Sciences

1. (a) Moving forward means $v(t) > 0$.

$$\text{From the graph: } v(t) = 0 \text{ at } t = 1, 5 \text{ and on } [8, 9], \quad v > 0 \text{ on } (1, 5) \text{ and } (9, 10).$$

$$\boxed{(1, 5) \text{ and } (9, 10)}$$

(b) At rest means $v(t) = 0$.

$$\boxed{t = 1,\ t = 5,\ \text{and } [8, 9]}$$

(c) The particle speeds up exactly when the velocity and acceleration have the same sign.

Interval	v	a	Speeding up?
$(0, 1)$	$-$	$+$	no
$(1, 2)$	$+$	$+$	yes
$(2, 4)$	$+$	0	no (constant speed)
$(4, 5)$	$+$	$-$	no
$(5, 6)$	$-$	$-$	yes
$(6, 8)$	$-$	$+$	no
$(8, 9)$	0	0	no
$(9, 10)$	$+$	$+$	yes

$$\boxed{\text{The particle is speeding up on } (1, 2),\ (5, 6),\ \text{and } (9, 10).}$$

2.

$$s(t) = 96t - 16t^2, \qquad v(t) = s'(t) = 96 - 32t.$$

(a) $v(1) = 96 - 32(1) = \boxed{64 \text{ ft/s}}$.

(b) Maximum height (Closed Interval Method):

1. *Find the time interval of motion.* The ball is in the air from launch until it returns to the ground. Set the height to zero: $s(t) = 96t - 16t^2 = 0 \Rightarrow t(6 - t) = 0$. Thus the motion occurs on the closed interval $[0, 6]$.
2. *Find critical points in* $(0, 6)$. Differentiate: $s'(t) = 96 - 32t$. Set $s'(t) = 0$: $96 - 32t = 0 \Rightarrow t = 3$. (This is the only critical number in $(0, 6)$.)
3. *Evaluate* $s(t)$ *at the endpoints and the critical point.*

$$s(0) = 0, \qquad s(3) = 96 \cdot 3 - 16 \cdot 3^2 = 288 - 144 = 144, \qquad s(6) = 96 \cdot 6 - 16 \cdot 6^2 = 576 - 576 = 0.$$

4. *Compare values.* The largest height on $[0, 6]$ is 144 ft, attained at $t = 3$ s.

$$\boxed{\text{Global maximum height} = 144 \text{ ft at } t = 3 \text{ s.}}$$

(c) Solve $s(t) = 128$:

$$96t - 16t^2 = 128 \iff -16\big(t^2 - 6t + 8\big) = 0 \iff (t-2)(t-4) = 0,$$
$$\Rightarrow\ t = 2 \text{ (up)},\quad t = 4 \text{ (down)}.$$

On the way up use $t = 2$:

$$v(2) = 96 - 32(2) = \boxed{32 \text{ ft/s (upward)}}.$$

3. (a) Quadrupling every hour with $P(0) = 80$ gives

$$\boxed{P(t) = 80 \cdot 4^{\,t}}.$$

(b) $P'(t) = 80 \cdot 4^{\,t} \ln 4$

$$\begin{aligned} P'(0.75) &= 80 \cdot 4^{3/4} \ln 4 \\ &= 80 \cdot 2^{3/2} \ln 4 \\ &= \boxed{80\sqrt{8}\,\ln 4 \ \approx\ 313 \text{ cells/hour}}. \end{aligned}$$

4.

$$C(q) = 5400 + 100\sqrt{q}, \qquad C'(q) = \frac{100}{2\sqrt{q}} = \frac{50}{\sqrt{q}}.$$

(i)

$$C'(400) = \frac{50}{\sqrt{400}} = \frac{50}{20} = \boxed{\$2.50 \text{ per gadget}}.$$

(ii) Interpretation: at $q = 400$, producing one more unit costs about $C'(400)$.

$$\boxed{\text{Estimated cost of the } 401^{\text{st}} \text{ gadget } \approx \$2.50.}$$

(iii)

$$\text{Actual marginal cost} = C(401) - C(400) = 100\Big(\sqrt{401} - 20\Big) = \boxed{100(\sqrt{401} - 20)\ \text{ dollars } \approx \$2.498}.$$

3.10 - Linear Approximation

1.

$$\text{Linearization formula: } L(x) = f(a) + f'(a)(x-a).$$

$$f(x) = \cos x, \quad a = \frac{\pi}{3}, \quad f(a) = \cos\frac{\pi}{3} = \frac{1}{2}.$$

$$f'(x) = -\sin x, \quad f'(a) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

$$\boxed{L(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right).}$$

2.

$$\text{Linearization formula: } L(x) = g(a) + g'(a)(x-a).$$

$$g(x) = \sqrt{4-x}, \quad a = 1, \quad g(a) = \sqrt{3}.$$

$$g'(x) = -\frac{1}{2\sqrt{4-x}}, \quad g'(a) = -\frac{1}{2\sqrt{3}}.$$

$$\boxed{L(x) = \sqrt{3} - \frac{1}{2\sqrt{3}}(x-1).}$$

3.

$$\text{Linearization formula: } L(x) = h(a) + h'(a)(x-a).$$

$$h(x) = e^{2x}, \quad a = 0, \quad h(a) = e^0 = 1.$$

$$h'(x) = 2e^{2x}, \quad h'(a) = 2.$$

$$\boxed{L(x) = 1 + 2x.}$$

4.

$$\text{Linearization at } a = 8: \quad L(x) = f(a) + f'(a)(x-a).$$

$$f(x) = \sqrt[3]{x} = x^{1/3}, \quad f(8) = 2, \quad f'(x) = \frac{1}{3}x^{-2/3}, \quad f'(8) = \frac{1}{12}.$$

$$\boxed{L(x) = 2 + \frac{1}{12}(x-8).}$$

Estimate at $x = 7.9$:

$$\sqrt[3]{7.9} = f(7.9) \approx L(7.9) = 2 + \frac{1}{12}(-0.1) = 2 - \frac{1}{120}.$$

Over/under? Since $f''(x) = -\frac{2}{9}x^{-5/3} < 0$ for $x > 0$, f is concave down near 8, so the tangent line lies above the curve. The linearization is an **overestimate** of $\sqrt[3]{7.9}$.

5.

$$\text{Linearization at } a = 0: \quad L(x) = g(a) + g'(a)(x - a).$$

$$g(x) = \sin x, \quad g(0) = 0, \quad g'(x) = \cos x, \quad g'(0) = 1.$$

$$\boxed{L(x) = x.}$$

Estimate at $x = 0.76$:

$$\sin(0.76) = g(0.76) \approx L(0.76) = 0.76.$$

Over/under? At $x = 0.76$, $\sin x$ is concave down ($g''(0.76) = -\sin(0.76) < 0$), so the tangent line is above the curve. The linearization is an **overestimate** of $\sin(0.76)$.

6.

$$\text{Linearization at } a = 2: \quad L(x) = h(a) + h'(a)(x - a).$$

$$h(x) = \ln x, \quad h(2) = \ln 2, \quad h'(x) = \frac{1}{x}, \quad h'(2) = \frac{1}{2}.$$

$$\boxed{L(x) = \ln 2 + \frac{1}{2}(x - 2).}$$

Estimate at $x = 2.05$:

$$\ln(2.05) = f(2.05) \approx L(2.05) = \ln 2 + \tfrac{1}{2}(0.05) = \ln 2 + 0.025 \approx 0.71815.$$

Over/under? Since $h''(x) = -\frac{1}{x^2} < 0$ on $(0, \infty)$, $\ln x$ is concave down, so the tangent line lies above the curve. The linearization is an **overestimate** of $\ln(2.05)$.

4.1 - Maximum and Minimum Values

1. **Critical numbers of** $f(x) = x^{\frac{2}{3}}(x-3)$.

$$f'(x) = \tfrac{2}{3}x^{-\frac{1}{3}}(x-3) + x^{\frac{2}{3}} = x^{-\frac{1}{3}}\left(\tfrac{2}{3}(x-3) + x\right) = x^{-\frac{1}{3}}\left(\tfrac{5}{3}x - 2\right).$$

   Critical numbers occur where $f' = 0$ or f' is undefined while f is defined:

$$\tfrac{5}{3}x - 2 = 0 \;\Rightarrow\; x = \tfrac{6}{5}, \qquad x^{-\frac{1}{3}} \text{ undefined at } x = 0 \text{ but } f(0) = 0 \;\Rightarrow\; x = 0 \text{ is critical.}$$

$$\boxed{\text{Critical numbers: } x = 0,\ \tfrac{6}{5}.}$$

2. **Critical numbers of** $f(x) = \dfrac{x^2 + 4x + 5}{x+1}$ $(x \neq -1)$.

$$f(x) = x + 3 + \frac{2}{x+1} \quad\Rightarrow\quad f'(x) = 1 - \frac{2}{(x+1)^2}.$$

   Solve $f'(x) = 0$ (note $x = -1$ not in domain):

$$1 - \frac{2}{(x+1)^2} = 0 \;\Rightarrow\; (x+1)^2 = 2 \;\Rightarrow\; x = -1 \pm \sqrt{2}.$$

$$\boxed{\text{Critical numbers: } x = -1-\sqrt{2},\ -1+\sqrt{2}.}$$

3. **Critical numbers of** $f(x) = \dfrac{x-1}{\sqrt{5-x}}$ $(x < 5)$.

$$f(x) = (x-1)(5-x)^{-1/2}, \quad f'(x) = (5-x)^{-1/2} + \frac{x-1}{2}(5-x)^{-3/2} = \frac{9-x}{2(5-x)^{3/2}}.$$

   Here $f'(x) = 0 \Rightarrow x = 9$ (not in domain), and f' is undefined at $x = 5$ (also not in domain).

$$\boxed{\text{No critical numbers on } (-\infty, 5).}$$

4. **Critical numbers of** $f(x) = x\,e^{-x^2}$.

$$f'(x) = e^{-x^2} + x \cdot (-2x)e^{-x^2} = e^{-x^2}(1 - 2x^2).$$

   Since $e^{-x^2} > 0$, set $1 - 2x^2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$.

$$\boxed{\text{Critical numbers: } x = \pm\tfrac{1}{\sqrt{2}}.}$$

5. **Absolute extrema of** $f(x) = x^3 - 3x$ on $[-2, 2]$ (closed-interval method).

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 0 \;\Rightarrow\; x = \pm 1.$$

Evaluate at endpoints and critical points:

$$f(-2) = -2, \quad f(-1) = 2, \quad f(1) = -2, \quad f(2) = 2.$$

$$\boxed{\text{Absolute max } = 2 \text{ at } x = -1, 2; \quad \text{Absolute min } = -2 \text{ at } x = -2, 1.}$$

6. **Absolute extrema of** $f(x) = \sqrt{9 - x^2}$ on $[-3, 2]$.

$$f'(x) = \frac{-x}{\sqrt{9 - x^2}} = 0 \Rightarrow x = 0 \text{ (interior)}.$$

Evaluate:

$$f(-3) = 0, \qquad f(0) = 3, \qquad f(2) = \sqrt{5}.$$

$$\boxed{\text{Absolute max } = 3 \text{ at } x = 0; \quad \text{Absolute min } = 0 \text{ at } x = -3.}$$

7. **Absolute extrema of** $f(x) = \ln x - \frac{x}{2}$ on $[1, 6]$.

$$f'(x) = \frac{1}{x} - \frac{1}{2} = 0 \Rightarrow x = 2 \text{ (interior)}.$$

Evaluate:

$$f(1) = -\tfrac{1}{2}, \qquad f(2) = \ln 2 - 1, \qquad f(6) = \ln 6 - 3.$$

Since $\ln 2 - 1 \approx -0.307$ and $\ln 6 - 3 \approx -1.208$,

$$\boxed{\text{Absolute max } = \ln 2 - 1 \text{ at } x = 2; \quad \text{Absolute min } = \ln 6 - 3 \text{ at } x = 6.}$$

8. **Absolute extrema of** $f(x) = (x - 1)^{2/3}$ on $[-2, 4]$.

$$f'(x) = \tfrac{2}{3}(x - 1)^{-1/3} \text{ (undefined at } x = 1), \quad f \text{ is defined at } x = 1.$$

Check endpoints and the nondifferentiable point:

$$f(-2) = 3^{2/3}, \qquad f(1) = 0, \qquad f(4) = 3^{2/3}.$$

$$\boxed{\text{Absolute max } = 3^{2/3} \text{ at } x = -2, 4; \quad \text{Absolute min } = 0 \text{ at } x = 1.}$$

4.2 - The Mean Value Theorem

1. *Check conditions.* f is continuous on $[2, 7]$ and differentiable on $(2, 7)$, so the Mean Value Theorem (MVT) applies.

   Average slope.

   $$\frac{f(7) - f(2)}{7 - 2} = \frac{-1 - 5}{5} = -\frac{6}{5}.$$

   MVT conclusion. There exists $c \in (2, 7)$ with

   $$\boxed{f'(c) = -\tfrac{6}{5}}.$$

2. *Check conditions.* f is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Also $f(0) = f(4) = 3$.

   Rolle's Theorem applies. Since the endpoint values agree, there exists $c \in (0, 4)$ with

   $$\boxed{f'(c) = 0}.$$

3. Let $s(t)$ be position (miles), continuous on $[0, 1.5]$ and differentiable on $(0, 1.5)$. Then the average speed is

   $$\frac{s(1.5) - s(0)}{1.5 - 0} = \frac{120 - 0}{1.5} = 80 \text{ mph}.$$

   MVT conclusion. There exists $c \in (0, 1.5)$ such that the instantaneous speed equals the average speed:

   $$\boxed{s'(c) = 80 \text{ mph, for some } c \in (0, 1.5)}.$$

4. *Given:* g is continuous on $[0, 6]$, differentiable on $(0, 6)$, and $g(0) = g(3) = g(6)$.

   Apply Rolle's Theorem twice.

   - On $[0, 3]$: $g(0) = g(3) \Rightarrow$ there exists $c_1 \in (0, 3)$ with $g'(c_1) = 0$.
   - On $[3, 6]$: $g(3) = g(6) \Rightarrow$ there exists $c_2 \in (3, 6)$ with $g'(c_2) = 0$.

   Thus there are at least two points with zero derivative:

   $$\boxed{\text{There is } c_1 \in (0, 3) \text{ and } c_2 \in (3, 6) \text{ such that } g'(c_1) = g'(c_2) = 0}.$$

5. *Secant through the endpoints.* With $f(0) = -1$ and $f(2) = 7$,

   $$\text{slope} = \frac{7 - (-1)}{2 - 0} = 4, \qquad \text{secant line: } y = -1 + 4x.$$

   MVT conclusion. Since f is continuous on $[0, 2]$ and differentiable on $(0, 2)$, there exists $c \in (0, 2)$ such that

   $$\boxed{f'(c) = 4}.$$

4.3 & 4.5 - Curve Sketching

1. $f(x) = x^3 - 6x^2 + 9x = x(x-3)^2$

   Derivatives. $f'(x) = 3(x-1)(x-3), \quad f''(x) = 6(x-2).$

   Critical numbers. $f'(x) = 0 \Rightarrow x = 1, 3.$

   Sign chart for f'.

interval	$(x-1)$	$(x-3)$	$f'(x)$
$(-\infty, 1)$	$-$	$-$	$+$
$(1, 3)$	$+$	$-$	$-$
$(3, \infty)$	$+$	$+$	$+$

   Increasing/Decreasing. Increasing on $(-\infty, 1), (3, \infty)$; decreasing on $(1, 3)$.

   Local extrema. Local max at $x = 1$ with $f(1) = 4$; local min at $x = 3$ with $f(3) = 0$.

   Sign chart for f''.

interval	$(x-2)$	$f''(x)$
$(-\infty, 2)$	$-$	$-$
$(2, \infty)$	$+$	$+$

   Concavity & inflection. Concave down on $(-\infty, 2)$, up on $(2, \infty)$. Inflection at $x = 2$ (since f'' changes sign), with $f(2) = 2$.

2. $f(x) = \dfrac{x}{x^2+4}$

   Derivatives. $f'(x) = \dfrac{(2-x)(2+x)}{(x^2+4)^2}$, $f''(x) = \dfrac{2x(x-2\sqrt{3})(x+2\sqrt{3})}{(x^2+4)^3}$.

   Critical numbers. Denominator > 0. $f'(x) = 0 \Rightarrow x = \pm 2$.

   Sign chart for f'.

interval	$(2-x)$	$(2+x)$	$(x^2+4)^2$	$f'(x)$
$(-\infty, -2)$	$+$	$-$	$+$	$-$
$(-2, 2)$	$+$	$+$	$+$	$+$
$(2, \infty)$	$-$	$+$	$+$	$-$

   Increasing/Decreasing. Increasing on $(-2, 2)$; decreasing on $(-\infty, -2), (2, \infty)$.

   Local extrema. At $x = -2$: $- \to +$ gives local min, $f(-2) = -\frac{1}{4}$. At $x = 2$: $+ \to -$ gives local max, $f(2) = \frac{1}{4}$.

   Sign chart for f''.

interval	x	$(x-2\sqrt{3})$	$(x+2\sqrt{3})$	$(x^2+4)^3$	$f''(x)$
$(-\infty, -2\sqrt{3})$	$-$	$-$	$-$	$+$	$-$
$(-2\sqrt{3}, 0)$	$-$	$-$	$+$	$+$	$+$
$(0, 2\sqrt{3})$	$+$	$-$	$+$	$+$	$-$
$(2\sqrt{3}, \infty)$	$+$	$+$	$+$	$+$	$+$

   Concavity & inflection. Concave down on $(-\infty, -2\sqrt{3}), (0, 2\sqrt{3})$; up on $(-2\sqrt{3}, 0), (2\sqrt{3}, \infty)$. Inflection at $x = -2\sqrt{3}$, 0, $2\sqrt{3}$.

3. $f(x) = (x+1)e^{-2x}$

Derivatives. $f'(x) = e^{-2x}(-2x-1),\ f''(x) = e^{-2x}(4x)$ (note $e^{-2x} > 0$).

Critical numbers. $f'(x) = 0 \Rightarrow -2x-1 = 0 \Rightarrow x = -\frac{1}{2}$.

Sign chart for f'.

interval	e^{-2x}	$(-2x-1)$	$f'(x)$
$(-\infty, -\frac{1}{2})$	$+$	$+$	$+$
$(-\frac{1}{2}, \infty)$	$+$	$-$	$-$

Increasing/Decreasing. Increasing on $(-\infty, -\frac{1}{2})$; decreasing on $(-\frac{1}{2}, \infty)$.

Local extrema. Local max at $x = -\frac{1}{2}$ with $f(-\frac{1}{2}) = \frac{1}{2}\,e$.

Sign chart for f''.

interval	e^{-2x}	x	$f''(x)$
$(-\infty, 0)$	$+$	$-$	$-$
$(0, \infty)$	$+$	$+$	$+$

Concavity & inflection. Concave down on $(-\infty, 0)$, up on $(0, \infty)$. Inflection at $x = 0$ with $f(0) = 1$.

4. $f(x) = x^5 - 10x^3 = x^3(x^2 - 10)$

Derivatives. $f'(x) = 5x^2(x-\sqrt{6})(x+\sqrt{6}),\ f''(x) = 20x(x-\sqrt{3})(x+\sqrt{3})$.

Critical numbers. $x = -\sqrt{6},\ 0,\ \sqrt{6}$.

Sign chart for f'.

interval	x^2	$(x-\sqrt{6})$	$(x+\sqrt{6})$	5	$f'(x)$
$(-\infty, -\sqrt{6})$	$+$	$-$	$-$	$+$	$+$
$(-\sqrt{6}, 0)$	$+$	$-$	$+$	$+$	$-$
$(0, \sqrt{6})$	$+$	$-$	$+$	$+$	$-$
$(\sqrt{6}, \infty)$	$+$	$+$	$+$	$+$	$+$

Increasing/Decreasing. Increasing on $(-\infty, -\sqrt{6}), (\sqrt{6}, \infty)$; decreasing on $(-\sqrt{6}, \sqrt{6})$.

Local extrema. Local max at $x = -\sqrt{6}$ with $f(-\sqrt{6}) = 4 \cdot 6^{3/2}$. No extremum at $x = 0$ (no sign change). Local min at $x = \sqrt{6}$ with $f(\sqrt{6}) = -4 \cdot 6^{3/2}$.

Sign chart for f''.

interval	x	$(x-\sqrt{3})$	$(x+\sqrt{3})$	20	$f''(x)$
$(-\infty, -\sqrt{3})$	$-$	$-$	$-$	$+$	$-$
$(-\sqrt{3}, 0)$	$-$	$-$	$+$	$+$	$+$
$(0, \sqrt{3})$	$+$	$-$	$+$	$+$	$-$
$(\sqrt{3}, \infty)$	$+$	$+$	$+$	$+$	$+$

Concavity & inflection. Concave down on $(-\infty, -\sqrt{3}), (0, \sqrt{3})$; up on $(-\sqrt{3}, 0), (\sqrt{3}, \infty)$. Inflection at $x = -\sqrt{3},\ 0,\ \sqrt{3}$.

5. $f(x) = x^4(x-2) = x^5 - 2x^4$

Derivatives. $f'(x) = x^3(5x-8),\ f''(x) = 4x^2(5x-6)$.

Critical numbers. $f'(x) = 0 \Rightarrow x = 0,\ \dfrac{8}{5}$.

Sign chart for f'.

interval	x^3	$(5x-8)$	$f'(x)$
$(-\infty, 0)$	$-$	$-$	$+$
$(0, \frac{8}{5})$	$+$	$-$	$-$
$(\frac{8}{5}, \infty)$	$+$	$+$	$+$

Increasing/Decreasing. Increasing on $(-\infty, 0), (\frac{8}{5}, \infty)$; decreasing on $(0, \frac{8}{5})$.

Local extrema. Local max at $x = 0$ with $f(0) = 0$; local min at $x = \frac{8}{5}$ with $f(\frac{8}{5}) = \left(\frac{8}{5}\right)^4\left(\frac{8}{5} - 2\right) = -\dfrac{8192}{3125}$.

Sign chart for f''.

interval	x^2	$(5x-6)$	$f''(x)$
$(-\infty, 0)$	$+$	$-$	$-$
$(0, \frac{6}{5})$	$+$	$-$	$-$
$(\frac{6}{5}, \infty)$	$+$	$+$	$+$

Concavity & inflection. Concave down on $(-\infty, \frac{6}{5})$; up on $(\frac{6}{5}, \infty)$. Inflection at $x = \frac{6}{5}$. (At $x = 0$, f'' has no sign change.)

6. **Examples with** $f'(0) = f''(0) = 0$.

   (i) Local maximum at 0: $f(x) = -x^4$.

$$f'(x) = -4x^3 \ \Rightarrow\ f'(0) = 0, \qquad f''(x) = -12x^2 \ \Rightarrow\ f''(0) = 0.$$

   Since $f(x) \le 0 = f(0)$ for all x, $x = 0$ is a (global) maximum.

   (ii) Inflection at 0 but no local extremum: $f(x) = x^3$.

$$f'(x) = 3x^2 \ \Rightarrow\ f'(0) = 0, \qquad f''(x) = 6x \ \Rightarrow\ f''(0) = 0.$$

   Here f'' changes sign at 0 (from < 0 to > 0), so $x = 0$ is an inflection point; f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$, hence no local max/min at 0.

7. $f''(x) = (x+3)^5(x-1)^4(x-5)^3$.

Zeros and multiplicities. -3 (odd 5), 1 (even 4), 5 (odd 3).

Sign chart for f''.

interval	$(x+3)^5$	$(x-1)^4$	$(x-5)^3$	$f''(x)$
$(-\infty, -3)$	$-$	$+$	$-$	$+$
$(-3, 1)$	$+$	$+$	$-$	$-$
$(1, 5)$	$+$	$+$	$-$	$-$
$(5, \infty)$	$+$	$+$	$+$	$+$

Concavity. Concave up on $(-\infty, -3)$ and $(5, \infty)$; concave down on $(-3, 5)$.

Inflection points. At $x = -3$ and $x = 5$ (odd multiplicity $\Rightarrow$ sign flips). No inflection at $x = 1$ (even multiplicity $\Rightarrow$ no sign change).

Multiplicity rule. At a zero of f'', an *odd* multiplicity flips the sign of f'' (concavity changes); an *even* multiplicity preserves the sign (no concavity change).

8. $f''(x) = (x-2)^2(x+1)^3(x-4)$.

   Zeros and multiplicities. 2 (even 2), -1 (odd 3), 4 (odd 1).

   Sign chart for f''.

interval	$(x-2)^2$	$(x+1)^3$	$(x-4)$	$f''(x)$
$(-\infty, -1)$	$+$	$-$	$-$	$+$
$(-1, 2)$	$+$	$+$	$-$	$-$
$(2, 4)$	$+$	$+$	$-$	$-$
$(4, \infty)$	$+$	$+$	$+$	$+$

   Concavity. Concave up on $(-\infty, -1)$ and $(4, \infty)$; concave down on $(-1, 4)$.

   Inflection points. $x = -1$ and $x = 4$.

9. $f''(x) = (x+2)(x-1)^2(x-3)$.

   Zeros and multiplicities. -2 (odd 1), 1 (even 2), 3 (odd 1).

   Sign chart for f''.

interval	$(x+2)$	$(x-1)^2$	$(x-3)$	$f''(x)$
$(-\infty, -2)$	$-$	$+$	$-$	$+$
$(-2, 1)$	$+$	$+$	$-$	$-$
$(1, 3)$	$+$	$+$	$-$	$-$
$(3, \infty)$	$+$	$+$	$+$	$+$

   Concavity. Concave up on $(-\infty, -2)$ and $(3, \infty)$; concave down on $(-2, 3)$.

   Inflection points. $x = -2$ and $x = 3$.

10.

11.

12.

4.4 - Indeterminate Forms and l'Hôpital's Rule

1. $\lim_{x\to 0} \frac{e^{2x}-1-2x}{x^2} \overset{0/0}{=} \lim_{x\to 0} \frac{2e^{2x}-2}{2x} \overset{0/0}{=} \lim_{x\to 0} \frac{4e^{2x}}{2} = \boxed{2}$.

2. $\lim_{x\to 0} \frac{\ln(1+x)-x+\frac{1}{2}x^2}{x^3} \overset{0/0}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}-1+x}{3x^2} \overset{0/0}{=} \lim_{x\to 0} \frac{-\frac{1}{(1+x)^2}+1}{6x} \overset{0/0}{=} \lim_{x\to 0} \frac{\frac{2}{(1+x)^3}}{6} = \boxed{\frac{1}{3}}$.

3.

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} &\overset{0/0}{=} \lim_{x\to 0} \frac{\sec^2 x - \cos x}{3x^2} \\
&\overset{0/0}{=} \lim_{x\to 0} \frac{2\sec^2 x \tan x + \sin x}{6x} \\
&\overset{0/0}{=} \lim_{x\to 0} \frac{4\sec^2 x \tan^2 x + 2\sec^4 x + \cos x}{6} \\
&= \boxed{\frac{1}{2}}.
\end{aligned}$$

4. $\lim_{x\to\infty} \frac{2x^2+3x+1}{x^2-5x+4} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{4x+3}{2x-5} = \lim_{x\to\infty} \frac{4+\frac{3}{x}}{2-\frac{5}{x}} = \boxed{2}$.

5. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = \boxed{0}$.

6. $\lim_{x\to\infty} \left(\sqrt{x^2+4x}-x\right) = \lim_{x\to\infty} \frac{x^2+4x-x^2}{\sqrt{x^2+4x}+x} = \lim_{x\to\infty} \frac{4x}{x\left(\sqrt{1+\frac{4}{x}}+1\right)} = \lim_{x\to\infty} \frac{4}{\sqrt{1+\frac{4}{x}}+1} = \boxed{2}$.

7. $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{(-\infty)/\infty}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = \boxed{0}$.

8. $\lim_{x\to 0^+} x\cot x = \lim_{x\to 0^+} \frac{x}{\tan x} \overset{0/0}{=} \lim_{x\to 0^+} \frac{1}{\sec^2 x} = \cos^2 0 = \boxed{1}$.

9. $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{2x}$

   Let $y = \left(1+\frac{3}{x}\right)^{2x} > 0$. Taking natural logs brings the exponent down:

$$\ln y = 2x\,\ln\!\left(1+\frac{3}{x}\right) = 2\cdot\frac{\ln\left(1+\frac{3}{x}\right)}{1/x}.$$

   As $x\to\infty$ this is a 0/0 form, so apply L'Hôpital directly in x:

$$\lim_{x\to\infty} \ln y = 2\cdot \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(1+3/x)}{\frac{d}{dx}(1/x)} = 2\cdot\lim_{x\to\infty} \frac{\frac{-3}{x^2(1+3/x)}}{-\frac{1}{x^2}} = 2\cdot\lim_{x\to\infty}\frac{3}{1+3/x} = 6.$$

   Therefore $y\to e^6$, i.e.

$$\boxed{\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x} = e^6}.$$

10. $\lim_{x\to 0^+} (\sin x)^x$

For $x > 0$ set $y = (\sin x)^x$. Then

$$\ln y = x \ln(\sin x) = \frac{\ln(\sin x)}{1/x}.$$

As $x \to 0^+$ this is $(-\infty)/\infty$, so use L'Hôpital:

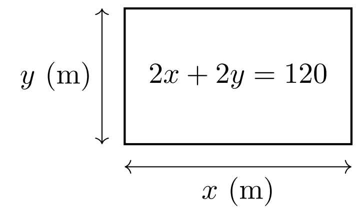
$$\begin{aligned}
\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(\sin x)}{(1/x)} &\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\cot x}{-1/x^2} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{-\csc^2 x}{2/x^3} \\
&= \lim_{x\to 0^+} \frac{x^3}{-2\sin^2 x} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{3x^2}{-4\sin x \cos x} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{6x}{-4(\cos^2 x - \sin^2 x)} \\
&= \frac{0}{-4} \\
&= \boxed{0}.
\end{aligned}$$

So $\ln y \to 0$ and $y \to e^0 = 1$. Therefore

$$\boxed{\lim_{x\to 0^+} (\sin x)^x = 1}.$$

4.7 - Applied Optimization

1. A rectangle has perimeter 120 m. Find the dimensions that maximize its area.



Let the width be x meters and the length be y meters. The perimeter condition $2x + 2y = 120$ gives $y = 60 - x$ with $0 \le x \le 60$. The area is a single–variable function

$$A(x) = xy = x(60 - x) = 60x - x^2.$$

This is a concave-down quadratic on the closed interval $[0, 60]$. Differentiate:

$$A'(x) = 60 - 2x \quad \Rightarrow \quad A'(x) = 0 \iff x = 30.$$

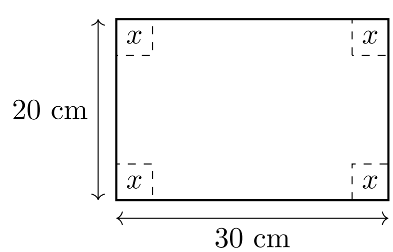
Evaluate at the endpoints and the critical point:

$$A(0) = 0, \qquad A(60) = 0, \qquad A(30) = 60(30) - 30^2 = 900.$$

Thus the maximum area occurs at $x = 30$ and $y = 60 - 30 = 30$.

The area is maximized by a 30 m $\times$ 30 m square (area 900 m^2).

2. An open-top box is made by cutting congruent squares of side x from each corner of a 20 cm $\times$ 30 cm sheet and folding up the sides. For what x is the volume maximized?



When squares of side x are removed and the sides are folded up, the height of the box is x, the base becomes $(30 - 2x)$ by $(20 - 2x)$, and the volume is

$$V(x) = x(30 - 2x)(20 - 2x) = x\big(600 - 100x + 4x^2\big) = 600x - 100x^2 + 4x^3.$$

Feasible values are $0 \le x \le 10$ (so that $20 - 2x \ge 0$); at the endpoints $x = 0, 10$ the volume is 0.

Differentiate to find interior critical points:

$$V'(x) = 600 - 200x + 12x^2 = 12x^2 - 200x + 600.$$

Solve $V'(x) = 0$ (divide by 2 if you like):

$$12x^2 - 200x + 600 = 0 \iff 3x^2 - 50x + 150 = 0 \implies x = \frac{50 \pm \sqrt{50^2 - 4 \cdot 3 \cdot 150}}{2 \cdot 3} = \frac{50 \pm 10\sqrt{7}}{6}.$$

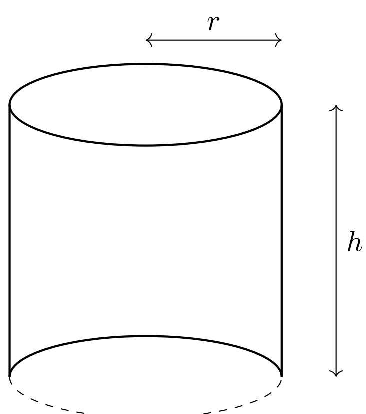
This gives $x = \dfrac{25 - 5\sqrt{7}}{3} \approx 3.92$ cm and $x = \dfrac{25 + 5\sqrt{7}}{3} \approx 12.74$ cm. The second value is outside the feasible interval $[0, 10]$, so we discard it.

To verify a maximum, evaluate the second derivative $V''(x) = 24x - 200$, which is negative at $x \approx 3.92$, or simply use the closed-interval method: $V(0) = 0$, $V(10) = 0$, and $V\left(\frac{25-5\sqrt{7}}{3}\right) > 0$. Therefore the volume is largest at

$$\boxed{x = \frac{25 - 5\sqrt{7}}{3} \text{ cm} \approx 3.92 \text{ cm}}.$$

At this cut size, the finished box has height $x \approx 3.92$ cm, base $(30 - 2x) \approx 22.15$ cm by $(20 - 2x) \approx 12.15$ cm, and volume about $V \approx 1.06 \times 10^3$ cm^3.

3. **Problem.** A right circular cylinder must hold 500 cm^3 of liquid. Find the radius r and height h that minimize its surface area.



Let r be the radius and h the height. The volume constraint is $\pi r^2 h = 500$, so $h = \dfrac{500}{\pi r^2}$ (with $r > 0$). The surface area of an open cylinder would differ, but for a *closed* cylinder (top and bottom) the area is

$$S = 2\pi r^2 + 2\pi r h.$$

Substitute the constraint into S to get a single-variable function:

$$S(r) = 2\pi r^2 + 2\pi r \left(\frac{500}{\pi r^2}\right) = 2\pi r^2 + \frac{1000}{r}, \qquad r > 0.$$

Differentiate and set to zero:

$$S'(r) = 4\pi r - \frac{1000}{r^2} = 0 \implies 4\pi r^3 = 1000 \implies r^3 = \frac{250}{\pi}.$$

Hence

$$r = \left(\frac{250}{\pi}\right)^{1/3} \approx 4.30 \text{ cm}.$$

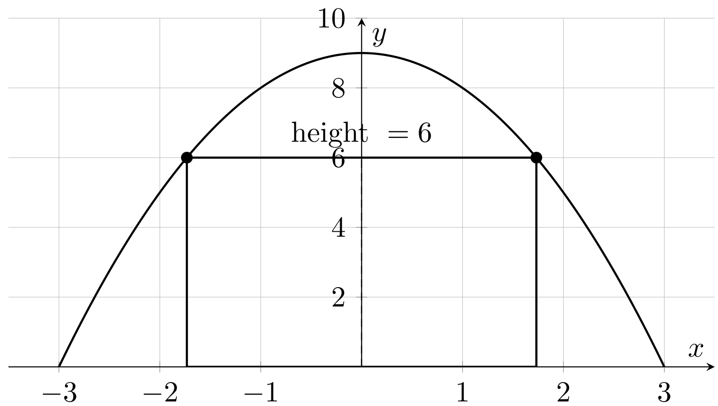
From the volume relation,

$$h = \frac{500}{\pi r^2} = \frac{500}{\pi} \cdot \frac{1}{\left(\frac{250}{\pi}\right)^{2/3}} = 2\left(\frac{250}{\pi}\right)^{1/3} = 2r \approx 8.60 \text{ cm}.$$

Because $S''(r) = 4\pi + \dfrac{2000}{r^3} > 0$ for $r > 0$, this critical point gives a (global) minimum.

$$\boxed{r = \left(\frac{250}{\pi}\right)^{1/3} \text{ cm}, \qquad h = 2\left(\frac{250}{\pi}\right)^{1/3} \text{ cm (so } h = 2r\text{).}}$$

4. **Problem.** A rectangle is inscribed in the region under $y = 9 - x^2$ and above the x-axis. Its base lies on the x-axis and its upper vertices touch the parabola. Find the dimensions that maximize the rectangle's area.



Because the region is symmetric about the y-axis, we take the rectangle to be symmetric as well. Let the right upper corner be at $(a,\ 9 - a^2)$ with $a \in [0, 3]$; then the left upper corner is $(-a,\ 9 - a^2)$. The base lies on the x-axis from $-a$ to a, so the width is $2a$ and the height is $9 - a^2$. Thus the area (as a single variable function of a) is

$$A(a) = (\text{width}) \times (\text{height}) = 2a\big(9 - a^2\big) = 18a - 2a^3, \qquad 0 \le a \le 3.$$

Differentiate and factor:

$$A'(a) = 18 - 6a^2 = 6(3 - a^2) = 6(\sqrt{3} - a)(\sqrt{3} + a).$$

Sign chart for A' (rows are intervals, columns are factors; note $\sqrt{3} + a > 0$ on $[0, 3]$):

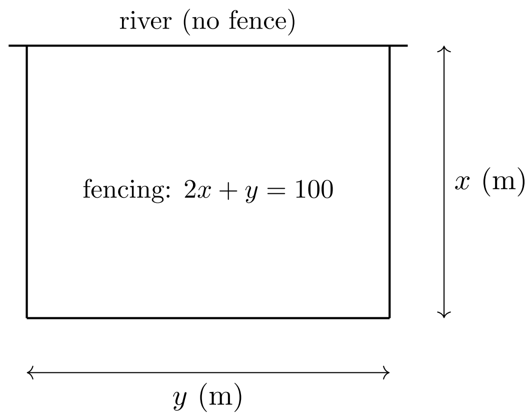
interval	$(\sqrt{3} - a)$	$(\sqrt{3} + a)$	$A'(a)$
$(0, \sqrt{3})$	$+$	$+$	$+$
$(\sqrt{3}, 3)$	$-$	$+$	$-$

So A increases on $(0, \sqrt{3})$ and decreases on $(\sqrt{3}, 3)$, giving a local maximum at $a = \sqrt{3}$. By the closed-interval method (or FDTAEV with end behavior), $A(0) = 0$ and $A(3) = 0$, so this local maximum is the *global* maximum on $[0, 3]$.

At $a = \sqrt{3}$ we have height $9 - a^2 = 9 - 3 = 6$ and width $2a = 2\sqrt{3}$. Therefore the maximizing rectangle is

$$\boxed{\text{width } = 2\sqrt{3}, \quad \text{height } = 6} \quad (\text{area } = 12\sqrt{3}).$$

5. **Problem.** A farmer has 100 m of fencing to enclose a rectangular pen against a straight river (*no fence along the river*). What dimensions maximize the area?



Let x be the depth (perpendicular to the river) and y be the length (along the river). Only three sides are fenced, so the fencing constraint is $2x + y = 100$, i.e. $y = 100 - 2x$ with $0 \leq x \leq 50$. The area is

$$A(x) = xy = x(100 - 2x) = 100x - 2x^2 \qquad (0 \leq x \leq 50).$$

Differentiate to locate interior extrema:

$$A'(x) = 100 - 4x = 4(25 - x).$$

Sign chart for $A'(x)$ (intervals as rows; factors as columns):

interval	$(25 - x)$	$A'(x)$
$(0, 25)$	$+$	$+$
$(25, 50)$	$-$	$-$

Thus A increases on $(0, 25)$ and decreases on $(25, 50)$; the derivative changes from $+$ to $-$ at $x = 25$, so $x = 25$ is a local maximum. By the *First Derivative Test for Absolute Extrema* on the closed interval $[0, 50]$, and since $A(0) = A(50) = 0$, this local maximum is the *global* maximum.

At $x = 25$, the length is $y = 100 - 2(25) = 50$. The maximal area is $A(25) = 25 \cdot 50 = 1250 \text{ m}^2$.

$$\boxed{\text{Optimal dimensions: } x = 25 \text{ m (depth)}, \quad y = 50 \text{ m (along the river).}}$$

6. **Problem.** Two positive numbers have product 64. Find the numbers that minimize their sum.

Let the numbers be $x > 0$ and $y > 0$ with $xy = 64$. Then $y = \dfrac{64}{x}$ and the sum is

$$S(x) = x + y = x + \frac{64}{x}, \qquad x > 0.$$

Differentiate:

$$S'(x) = 1 - \frac{64}{x^2} = \frac{x^2 - 64}{x^2} = \frac{(x-8)(x+8)}{x^2}.$$

On the domain $x > 0$, both $x^2 > 0$ and $x + 8 > 0$, so the sign of $S'(x)$ is controlled by $(x - 8)$. The sign chart (intervals as rows, factors as columns) is:

interval	x^2	$(x-8)$	$(x+8)$	$S'(x)$
$(0, 8)$	$+$	$-$	$+$	$-$
$(8, \infty)$	$+$	$+$	$+$	$+$

Thus S is decreasing on $(0, 8)$ and increasing on $(8, \infty)$; the derivative changes from negative to positive at $x = 8$, so $x = 8$ is a local minimum. Since this is the only critical point on $(0, \infty)$, this local minimum is the *global* minimum (FDTAEV) on this interval.

At $x = 8$, $y = \dfrac{64}{8} = 8$, and the minimal sum is $S(8) = 8 + 8 = 16$.

$$\boxed{\text{The sum is minimized at } (x, y) = (8, 8) \text{ with sum 16.}}$$

7. **Problem.** A shop sells a gadget for \$60 and sells 40 units per week. For each \$3 decrease in price, weekly sales increase by 8 units. What price maximizes weekly *revenue*? Justify.

Let x denote the weekly quantity sold and $p(x)$ the price (in dollars) at that quantity. From the statement: whenever the price decreases by \$3, the quantity increases by 8.

Table (build the linear demand from the given trend):

x (units)	40	48	56	64
$p(x)$ (\$)	60	57	54	51

Each time x goes up by 8, p goes down by 3. Thus the slope is $\Delta p/\Delta x = -3/8$. Using the point $(x, p) = (40, 60)$, the demand line is

$$p(x) = 60 + \left(-\frac{3}{8}\right)(x - 40) = 75 - \frac{3}{8}x.$$

(Checks: $x = 40 \Rightarrow p = 60$; if $x = 0$, then $p = 75$; if $p = 0$, then $x = 200$. Hence a natural domain is $0 \le x \le 200$.)

Revenue as a function of quantity. $R(x) = x\,p(x) = x\big(75 - \frac{3}{8}x\big) = 75x - \frac{3}{8}x^2$, for $0 \le x \le 200$.

Differentiate and make a sign chart (FDTAEV on [0,200]**).**

$$R'(x) = 75 - \frac{3}{4}x = \frac{3}{4}(100 - x).$$

interval	$(100 - x)$	$R'(x)$
$(0, 100)$	$+$	$+$
$(100, 200)$	$-$	$-$

So R increases on $(0, 100)$ and decreases on $(100, 200)$. Because R is continuous on $[0, 200]$ and $R(0) = R(200) = 0$, the First Derivative Test for Absolute Extrema on this closed interval shows the interior critical point $x = 100$ gives the *absolute* maximum.

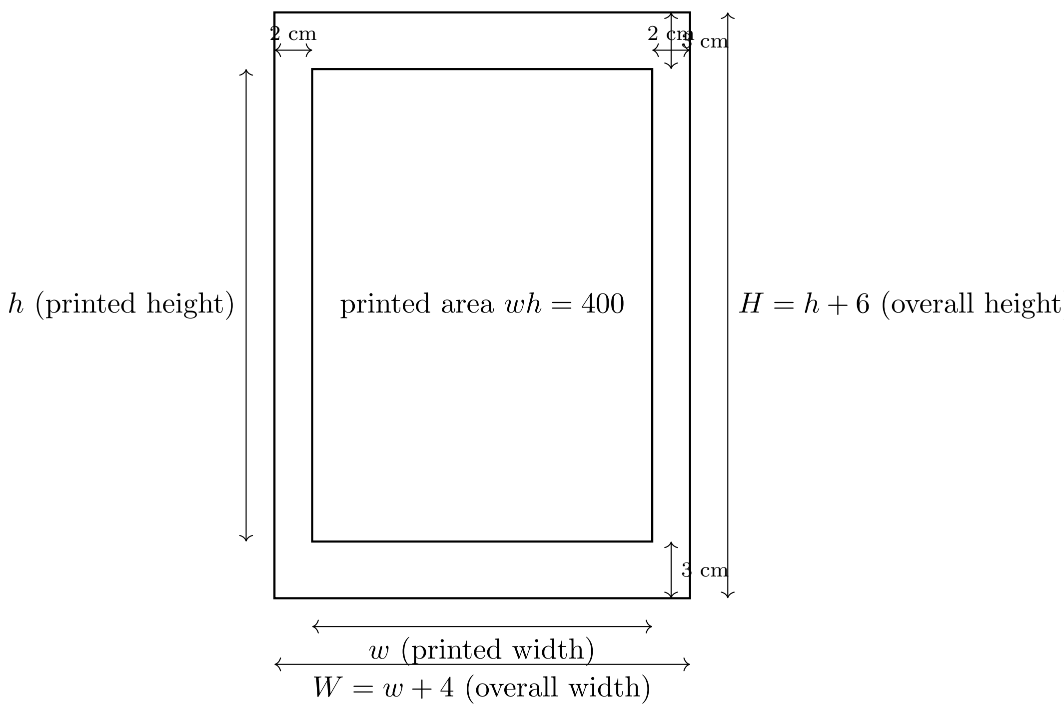
At $x = 100$, the price is

$$p(100) = 75 - \frac{3}{8} \cdot 100 = 75 - 37.5 = 37.5 \text{ (dollars)},$$

and the maximum weekly revenue is $R_{\max} = 100 \times 37.5 = \$3{,}750$.

$$\boxed{\text{Optimal price } = \$37.50, \quad \text{quantity } = 100 \text{ units/week}, \quad R_{\max} = \$3{,}750.}$$

8. **Problem.** A poster must contain $400\,\text{cm}^2$ of printed text. It has margins of 3 cm at the top and bottom and 2 cm on each side. Find the overall dimensions that minimize the total paper area.



Let the printed rectangle have width w cm and height h cm. The margin requirements make the overall width and height

$$W = w + 4, \qquad H = h + 6,$$

so the total paper area is $A_{\text{paper}} = WH = (w+4)(h+6)$. Because the printed area must be $wh = 400$, we have $h = \dfrac{400}{w}$ with $w > 0$. Substitute into A_{paper} to get a single-variable function:

$$A(w) = (w + 4)\left(\frac{400}{w} + 6\right) = \underbrace{6w + \frac{1600}{w}}_{\text{depends on } w} + 424, \qquad w > 0.$$

First derivative and sign chart (FDTAEV).

$$A'(w) = 6 - \frac{1600}{w^2} = \frac{6w^2 - 1600}{w^2} = \frac{6\left(w - \frac{40}{\sqrt{6}}\right)\left(w + \frac{40}{\sqrt{6}}\right)}{w^2}.$$

On the domain $w > 0$, both $w^2 > 0$ and $\left(w + \frac{40}{\sqrt{6}}\right) > 0$. Thus the sign of $A'(w)$ is set by $\left(w - \frac{40}{\sqrt{6}}\right)$.

interval	w^2	$\left(w - \frac{40}{\sqrt{6}}\right)$	$\left(w + \frac{40}{\sqrt{6}}\right)$	$A'(w)$
$(0, \frac{40}{\sqrt{6}})$	+	−	+	−
$(\frac{40}{\sqrt{6}}, \infty)$	+	+	+	+

So A decreases on $\left(0, \frac{40}{\sqrt{6}}\right)$ and increases on $\left(\frac{40}{\sqrt{6}}, \infty\right)$. By the *First Derivative Test for Absolute Extrema*, the unique critical point

$$w = \frac{40}{\sqrt{6}}$$

gives the *global* minimum of the total paper area.

At this w,

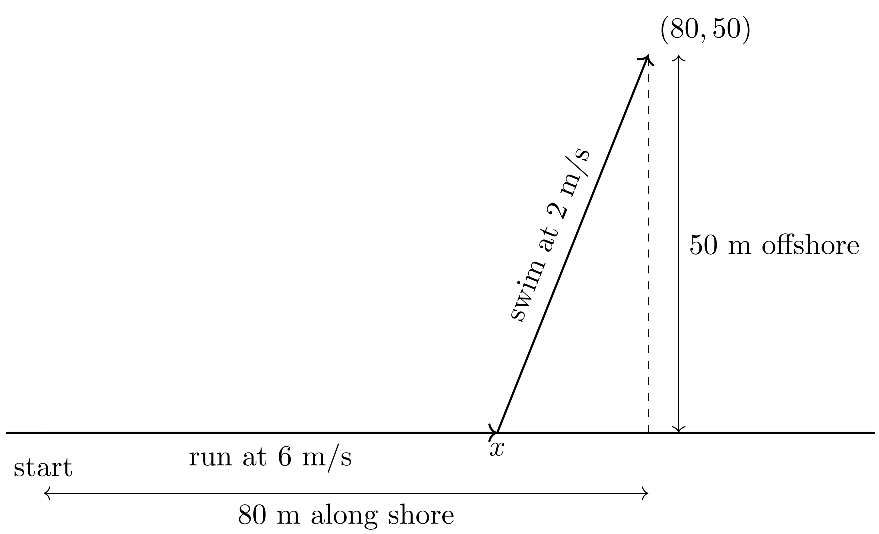
$$h = \frac{400}{w} = \frac{400}{40/\sqrt{6}} = 10\sqrt{6}.$$

Therefore the optimal *overall* dimensions are

$$W = w + 4 = \frac{40}{\sqrt{6}} + 4 \approx 20.33 \text{ cm}, \qquad H = h + 6 = 10\sqrt{6} + 6 \approx 30.50 \text{ cm}.$$

$$\boxed{\text{Overall size } \approx 20.33 \text{ cm by } 30.50 \text{ cm}; \quad \text{printed } w = \frac{40}{\sqrt{6}} \text{ cm}, \ h = 10\sqrt{6} \text{ cm}.}$$

9. **Problem.** A lifeguard can run at 6 m/s along the beach and swim at 2 m/s. A swimmer is 50 m offshore and 80 m down the beach from the lifeguard's starting point. Where should the lifeguard enter the water to *minimize time*?



Let the lifeguard run x meters along the shore before entering the water ($0 \le x \le 80$). Then:

$$\text{run distance } = x, \qquad \text{swim distance } = \sqrt{(80-x)^2 + 50^2}.$$

Total time as a function of x:

$$T(x) = \frac{\text{run}}{\text{run speed}} + \frac{\text{swim}}{\text{swim speed}} = \frac{x}{6} + \frac{\sqrt{(80-x)^2+50^2}}{2}, \qquad 0 \le x \le 80.$$

Differentiate and find the critical point.

$$T'(x) = \frac{1}{6} - \frac{80-x}{2\sqrt{(80-x)^2+50^2}}.$$

Set $T'(x) = 0$:

$$\frac{80-x}{\sqrt{(80-x)^2+50^2}} = \frac{1}{3}.$$

Square and solve:

$$(80-x)^2 = \frac{1}{9}\big((80-x)^2 + 50^2\big) \implies 8(80-x)^2 = 2500 \implies 80 - x = \frac{25}{\sqrt{2}}.$$

Therefore

$$x = 80 - \frac{25}{\sqrt{2}} \approx 62.3 \text{ m}.$$

Sign chart for $T'(x)$ (FDTAEV on the closed interval $[0, 80]$).

For $0 < x < 80 - \frac{25}{\sqrt{2}}$, the fraction $\dfrac{80-x}{2\sqrt{(80-x)^2+50^2}} > \dfrac{1}{6}$, so $T'(x) < 0$ (time decreases). For $80 - \frac{25}{\sqrt{2}} < x < 80$, the fraction $< \dfrac{1}{6}$, so $T'(x) > 0$ (time increases). Also T is continuous on $[0, 80]$ with

$$T(0) = \frac{\sqrt{80^2+50^2}}{2} \approx 47.2 \text{ s}, \qquad T(80) = \frac{80}{6} + \frac{50}{2} \approx 38.3 \text{ s}.$$

By the *First Derivative Test for Absolute Extrema* on $[0, 80]$, the unique interior point where T' changes from negative to positive gives the *global minimum*.

$$\boxed{\text{Enter the water about } x = 62.3 \text{ m down the beach from the start}}$$

10. **Problem.** A 100 cm piece of wire is cut into two pieces: one bent into a circle, the other into a square. How should the cut be made to *minimize* the total area? (Also: how to *maximize* it?)

circle wire $= x$ cm   square wire $= 100 - x$ cm

Let x cm go to the *circle*, so $100 - x$ cm go to the *square*. For the circle, $x = 2\pi r \Rightarrow r = \dfrac{x}{2\pi}$ and the area is

$$A_{\text{circ}} = \pi r^2 = \frac{x^2}{4\pi}.$$

For the square, $100 - x = 4s \Rightarrow s = \dfrac{100 - x}{4}$ and the area is

$$A_{\text{sq}} = s^2 = \frac{(100 - x)^2}{16}.$$

Total area (on the closed interval $0 \le x \le 100$) is

$$A(x) = \frac{x^2}{4\pi} + \frac{(100 - x)^2}{16}.$$

Differentiate and use a sign chart (FDTAEV).

$$A'(x) = \frac{x}{2\pi} - \frac{100 - x}{8} = \frac{(4 + \pi)x - 100\pi}{8\pi}.$$

The denominator $8\pi > 0$; the sign is set by $(4 + \pi)x - 100\pi$. The unique critical point is

$$(4 + \pi)x - 100\pi = 0 \quad \Longrightarrow \quad x^* = \frac{100\pi}{\pi + 4}.$$

interval	$(4+\pi)x - 100\pi$	$A'(x)$
$(0,\ x^*)$	$-$	$-$
$(x^*,\ 100)$	$+$	$+$

Thus A decreases on $(0, x^*)$ and increases on $(x^*, 100)$; by the *First Derivative Test for Absolute Extrema* on the closed interval $[0, 100]$, x^* gives the *global minimum*. At the endpoints,

$$A(0) = \frac{100^2}{16} = 625, \qquad A(100) = \frac{100^2}{4\pi} = \frac{2500}{\pi} \approx 795.8.$$

Therefore the *maximum* occurs at an endpoint; since $A(100) > A(0)$, the maximum is at $x = 100$ (all wire to the circle).

Answer (dimensions).

$$\boxed{\text{Minimum area when } x = \frac{100\pi}{\pi + 4} \text{ cm goes to the circle, and } 100 - x = \frac{400}{\pi + 4} \text{ cm to the square.}}$$

The corresponding radius and square side are

$$r = \frac{x}{2\pi} = \frac{50}{\pi + 4}, \qquad s = \frac{100 - x}{4} = \frac{100}{\pi + 4} \quad (\text{so } s = 2r).$$

$$\boxed{\text{Maximum area at } x = 100 \text{ cm (all circle), giving } A_{\max} = \frac{2500}{\pi} \text{ cm}^2.}$$

(At $x = 0$ (all square) the area is 625 cm^2, which is smaller since $\dfrac{2500}{\pi} > 625$.)

4.9 - Antiderivatives

1. $f(x) = 3x^2 - 4x + \cos x.$

$$\text{Use } (x^3)' = 3x^2,\ (-2x^2)' = -4x,\ (\sin x)' = \cos x.$$
$$\boxed{F(x) = x^3 - 2x^2 + \sin x + C.}$$

2. $f(x) = e^{2x} - 5.$

$$\text{Use } \left(\tfrac{1}{2}e^{2x}\right)' = e^{2x},\ (-5x)' = -5.$$
$$\boxed{F(x) = \tfrac{1}{2}e^{2x} - 5x + C.}$$

3. $f(x) = \sec^2 x - 2x.$

$$\text{Use } (\tan x)' = \sec^2 x,\ (-x^2)' = -2x.$$
$$\boxed{F(x) = \tan x - x^2 + C.}$$

4. $f(x) = x^{-3/2} + 5x^{1/2}$ (take $x > 0$).

$$\text{Use } \left(-2x^{-1/2}\right)' = x^{-3/2},\ \left(\tfrac{10}{3}x^{3/2}\right)' = 5x^{1/2}.$$
$$\boxed{F(x) = -2x^{-1/2} + \tfrac{10}{3}x^{3/2} + C.}$$

5. $f(x) = \dfrac{1}{x} + \dfrac{1}{1+x^2}.$

$$\text{Use } (\ln|x|)' = \tfrac{1}{x},\ (\arctan x)' = \tfrac{1}{1+x^2}.$$
$$\boxed{F(x) = \ln|x| + \arctan x + C.}$$

6. $f(x) = \cos(3x) - 4\sin(2x).$

$$\text{Use } \left(\tfrac{1}{3}\sin(3x)\right)' = \cos(3x),\ \left(2\cos(2x)\right)' = -4\sin(2x).$$
$$\boxed{F(x) = \tfrac{1}{3}\sin(3x) + 2\cos(2x) + C.}$$

7. $F'(x) = 2x\,e^{x^2},\ F(0) = 5.$

$$\text{Since } (e^{x^2})' = 2x\,e^{x^2},\quad F(x) = e^{x^2} + C.$$
$$F(0) = 5:\ 1 + C = 5 \Rightarrow C = 4.$$
$$\boxed{F(x) = e^{x^2} + 4.}$$

8. $F'(x) = \sqrt{x} + \dfrac{1}{x},\ F(1) = 0\ \ (\text{take } x > 0).$

$$F(x) = \tfrac{2}{3}x^{3/2} + \ln x + C.$$
$$F(1) = 0:\ \tfrac{2}{3} + 0 + C = 0 \Rightarrow C = -\tfrac{2}{3}.$$
$$\boxed{F(x) = \tfrac{2}{3}x^{3/2} + \ln x - \tfrac{2}{3}.}$$

9. $F'(x) = \dfrac{1}{\sqrt{1-x^2}}$ on $(-1,1),\ F(0) = 0.$

$$\text{Since } (\arcsin x)' = \frac{1}{\sqrt{1-x^2}},\quad F(x) = \arcsin x + C.$$
$$F(0) = 0:\ C = 0.$$
$$\boxed{F(x) = \arcsin x \quad (-1 < x < 1).}$$

10. $F'(x) = e^{-x} - 3x^2 + 2,\ F(2) = 7.$

$$F(x) = -e^{-x} - x^3 + 2x + C.$$
$$F(2) = 7:\ -e^{-2} - 8 + 4 + C = 7 \Rightarrow C = 11 + e^{-2}.$$
$$\boxed{F(x) = -e^{-x} - x^3 + 2x + 11 + e^{-2}.}$$