Midterm 2 Study Guide

MATH1300 - Calculus I

Fall 2025

Contents

2.8 - The Derivative as a Function

The derivative is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

and gives, for each x, the slope of the tangent line to $y = f(x)$ (the instantaneous rate of change). The domain of f' consists of the points where this limit exists and is often smaller than the domain of f. If f is differentiable at a, then f is continuous at a; non-differentiability typically occurs at corners/cusps, discontinuities, or vertical tangents.

1. Let
$$L = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}.$$
Find a function $f(x)$ such that $L = f'(x)$.

2. Let $g(x) = \cos x + 5$. Use the limit definition of the derivative to write $g'(x)$ at a general x.

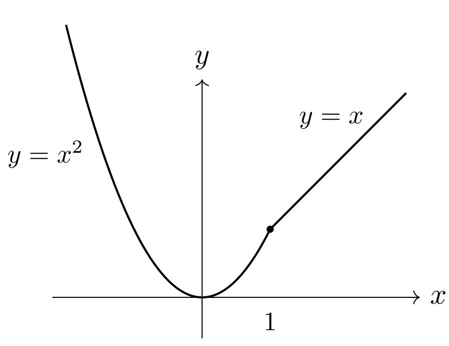
3. Compute the following limit:
$$\lim_{h \to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \sin\left(\frac{\pi}{6}\right)}{h}.$$

4. Compute the following limit:
$$\lim_{x \to 2} \frac{x^5 - 4x - \left(2^5 - 4 \cdot 2\right)}{x - 2}.$$

5. Let
$$f(x) = \begin{cases} ax^2 + b, & x < 2, \\ 3x + 1, & x \geq 2. \end{cases}$$
Find constants a, b so that f is *differentiable* at $x = 2$.

6. For each function, determine whether it is differentiable at 0. If yes, find $f'(0)$.
   (a) $g(x) = |x|^{2/3}$
   (b) $h(x) = x|x|$.

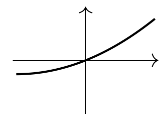
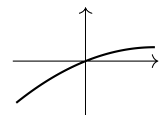
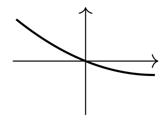
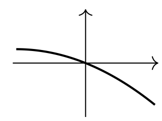
7.



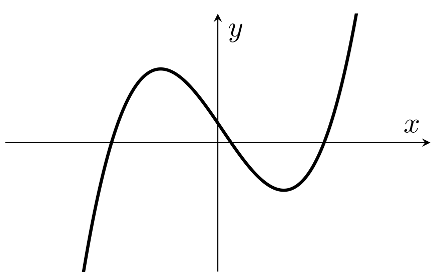
(a) Is f continuous at $x = 1$?
(b) Compute $f'(x)$ for $x < 1$ and for $x > 1$.
(c) Is f differentiable at $x = 1$? Justify using one–sided slopes.

4.3 - What f' and f'' say about f

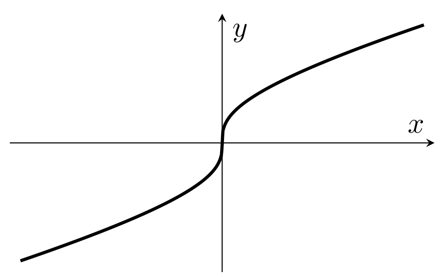
The sign of the first derivative $f'(x)$ tells whether f is increasing or decreasing, and $|f'(x)|$ reflects how steep the graph is. The second derivative $f''(x)$ controls concavity: $f'' > 0$ means concave up, and $f'' < 0$ means concave down.

$f'(x)$	$f''(x)$	**What f is doing**	**Schematic of $f(x)$**
> 0	> 0	Increasing; concave up (slopes becoming *more positive*).	
> 0	< 0	Increasing; concave down (slopes becoming *less positive*).	
< 0	> 0	Decreasing; concave up (slopes becoming *less negative*).	
< 0	< 0	Decreasing; concave down (slopes becoming *more negative*).	

1. For each of the following graphs of $f(x)$, sketch the graph of $f'(x)$.



(i)



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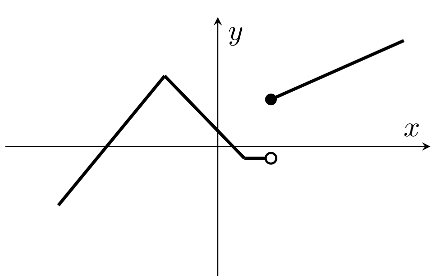

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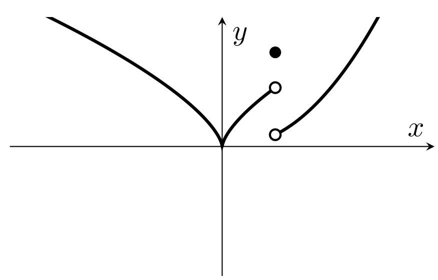


(iv)

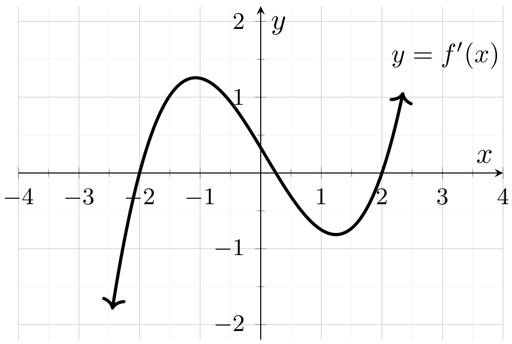
2. Each panel shows $y = f'(x)$. Estimate the following for each of the corresponding $f(x)$:

   (a) Where is f increasing?

   (b) Where is f decreasing?

   (c) Where is f concave up?

   (d) Where is f concave down?



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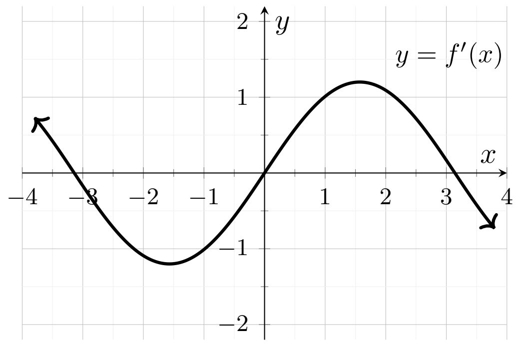

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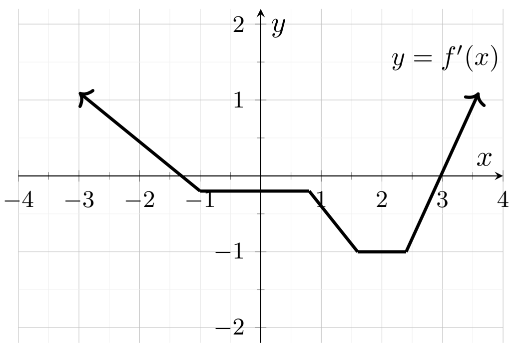

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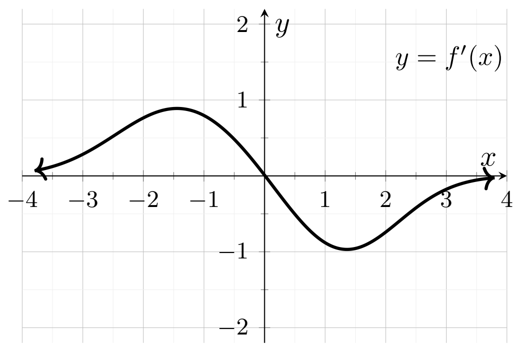

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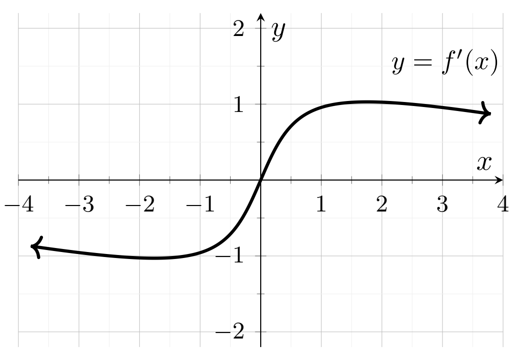

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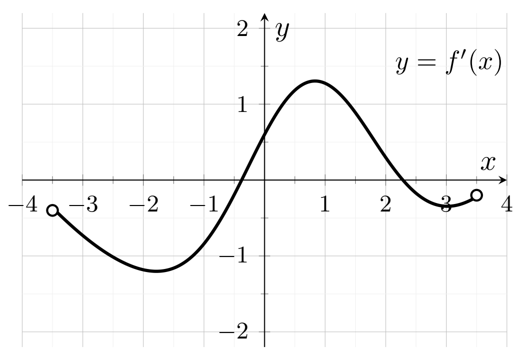

(vi)

3.1 - Derivatives of Polynomials and Exponential Functions

In this section, we learn the basic rules for differentiating constant functions, power functions, polynomials, and exponential functions. These rules form the foundation for nearly all differentiation techniques in calculus.

Constant Function

$$\frac{d}{dx}(c) = 0.$$

Constant Multiple

$$\big(c\, f(x)\big)' = c\, f'(x).$$

Sum Rule

$$\big(f(x) + g(x)\big)' = f'(x) + g'(x).$$

Difference Rule

$$\big(f(x) - g(x)\big)' = f'(x) - g'(x).$$

Power Rule

$$\frac{d}{dx}\big(x^n\big) = n\, x^{n-1} \quad (n \in \mathbb{R}).$$

Natural Exponential

$$\frac{d}{dx}\big(e^x\big) = e^x.$$

1. Find the derivative: $f(x) = x^{54} - 2x + 10$

2. Find the derivative: $g(x) = \frac{1}{5}x^3 - 9x + 20$

3. Find the derivative: $f(t) = 12e^t$

4. Find the derivative: $F(t) = t^6 - e^4$

5. Find the derivative: $r(z) = z^{-\frac{3}{2}} + z^{-5}$

6. Find the derivative: $g(x) = \dfrac{2}{\sqrt{x}} - \dfrac{3}{x^4}$

7. Find the derivative: $f(x) = x^2(x-5)^2$

8. Find the derivative: $y = \dfrac{7+4x}{2x^3}$

3.2 - The Product and Quotient Rules

In this section we extend our derivative formulas to products and quotients of functions. The rules below combine with the constant, power, and exponential rules to handle a wide range of functions.

Product Rule

Leibniz Notation	**Prime Notation**
$\dfrac{d}{dx}\big(uv\big) = u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$	$\big(f(x)g(x)\big)' = f(x)\,g'(x) + g(x)\,f'(x)$

Quotient Rule

Leibniz Notation	**Prime Notation**
$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$	$\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{\big(g(x)\big)^2}$

1. Differentiate: $y = (2x^2 - 5)(x^3 + 4x)$
2. Differentiate: $f(x) = x^2 e^{2x}$
3. Differentiate: $g(x) = \dfrac{5x}{e^x}$
4. Differentiate: $h(t) = \dfrac{t^2 - 3}{t^3 + 1}$
5. Differentiate $y = \dfrac{1 + x}{3 + e^x}$
6. Differentiate: $y = \dfrac{x^2 e^x}{1 + x}$
7. Find an equation of the tangent line to the curve $y = \dfrac{x^2}{4 + x^2}$ at the point $(2, \frac{1}{2})$
8. Suppose $f(2) = 3,\ f'(2) = -1,\ g(2) = -4,\ g'(2) = 5$. Compute:

$$(fg)'(2), \qquad \left(\frac{f}{g}\right)'(2), \qquad \left(\frac{g}{f}\right)'(2).$$

3.3 - Derivatives of Trig Functions

In this section we differentiate the six basic trigonometric functions. The formulas are:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x\,\tan x \qquad \frac{d}{dx}\csc x = -\csc x\,\cot x$$

1. Differentiate $p(t) = 4\cos t - 3\sin t$.

2. Differentiate $y = \sec x - 2\tan x$.

3. Evaluate $\dfrac{d}{du}\left(u^3 + \cot u\right)$.

4. Differentiate $h(\theta) = \theta^2 \sin\theta\cos\theta$.

5. Differentiate $y = \cot z + \csc z$.

6. Find the 205^{th} derivative of $\sin x$.

7. Find the 1900^{th} derivative of $\cos x$.

8. Compute $\dfrac{d}{dx}(e^x \csc x)$.

9. Compute $\dfrac{d}{ds}\left(\sec s\,\tan s\right)$.

10. Find $f'(x)$ if $f(x) = \dfrac{\tan x}{1 + \tan^2 x}$.

11. Find $g'(x)$ if $g(x) = \dfrac{\cos x}{1 - \sin x}$.

12. Find the tangent line to $y = e^x \sin x + \cos x$ at the point $(0, 1)$.

13. Find the tangent line to $y = \csc x$ at $x = \dfrac{\pi}{3}$.

14. Find all x in $[0, 2\pi]$ where the tangent to $g(x) = \sin x + \cos x$ is parallel to the line $y = -x$.

15. Find the x in $(-\frac{\pi}{2}, \frac{\pi}{2})$ where the slope of the tangent to $y = \tan x$ equals 3.

3.4 - The Chain Rule

In this section, we differentiate compositions of functions. To do so, we use the Chain Rule:

Leibniz Notation	**Prime Notation**
$\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$	$\big(f(g(x))\big)' = f'\big(g(x)\big)\,g'(x).$

1. Differentiate $y = \sin\big((3x-1)^4\big)$.
2. Differentiate $y = \big(\cos(2x-1)\big)^5$.
3. Differentiate $y = \sqrt{1+\cot(x^2-x)}$.
4. Differentiate $y = 3^{2x-\sin x}$.
5. Differentiate $y = \big(1+e^{x^3}\big)^7$.
6. Differentiate $y = e^{\tan\left(\frac{1}{\sqrt{x}}\right)}$ for $x > 0$.
7. Differentiate $y = \sqrt{1-\tan(2x)}$.
8. Differentiate $y = x^2 5^{\cos x}$.
9. Differentiate $y = \dfrac{7^{\tan x}}{\sin x}$.
10. Differentiate $y = \dfrac{e^{x^2}}{(1+x^4)^{3/2}}$.
11. Differentiate $y = \big(\sec x \cdot \csc x\big)^4$.
12. Let f, g, h be differentiable. Assume
$$f(1) = 2, \quad f'(1) = -3, \quad g(2) = -4, \quad g'(2) = 5, \quad h(-4) = 1, \quad h'(-4) = 7.$$
Define $H(x) = h\big(g(f(x))\big)$. Find $H'(1)$.
13. Let f, g be differentiable. Assume
$$f(2) = 1, \quad f'(2) = -5, \quad g(1) = 3, \quad g'(1) = 4.$$
Define $G(x) = g\big(f(x^3)\big)$. Evaluate $G'(\sqrt[3]{2})$.
14. Find the equation of the tangent line to $y = \dfrac{\sqrt{1+\cos x}}{1+x^2}$ at $x = 0$.
15. Find all x in $[0, 2\pi]$ where the tangent to $y = e^{-x}\sin x$ is horizontal.

3.5 - Implicit Differentiation

When a curve is defined implicitly by an equation relating x and y, we can still find slopes of tangent lines by treating y as a function $y(x)$ near our points of interest. For more details about why this is possible, see the Implicit Function Theorem.

Implicit Derivative Process

1. **Start from the relation.** Write the given equation relating the variables x and y.
2. **Differentiate both sides with respect to** $\boldsymbol{x}$**.** Apply derivative rules as needed.
   Every time you differentiate a term involving y, multiply by a factor $y' = \dfrac{dy}{dx}$ via the Chain Rule.
3. **Collect** $\boldsymbol{y'}$**-terms.** Move all terms containing y' to one side and all other terms to the opposite side.
4. **Factor and solve for** $\boldsymbol{y'}$**.** Factor out y' and isolate it to obtain an explicit formula $y' = \dfrac{dy}{dx}$ in terms of x and y.

Derivative Patterns (with $y = y(x)$)

$$\frac{d}{dx}[y] = y'$$
$$\frac{d}{dx}[y^n] = n\,y^{n-1}y' \quad (n \in \mathbb{R})$$
$$\frac{d}{dx}[\sin y] = \cos y\, y'$$
$$\frac{d}{dx}[\cos y] = -\sin y\, y'$$
$$\frac{d}{dx}[\tan y] = \sec^2 y\, y'$$
$$\frac{d}{dx}[e^y] = e^y\, y'$$
$$\frac{d}{dx}[a^y] = a^y \ln(a)\, y'$$

Find $\frac{dy}{dx}$ for each implicitly defined curve.

1. $x^2 + xy + y^2 = 7$
2. $x^3 + y^3 = 6xy$
3. $x^2y + xy^2 = 6$
4. $\sin(x+y) = xy$
5. $\cos(xy) + y = x$
6. $e^{x+y} = x^2 - y$
7. $x^2 + y^2 = \sin(xy)$
8. $\tan y = x\,e^y$

Compute $\frac{dy}{dx}$ at the indicated point.

9. $x^2y + y^2 = 4$ at $(0,2)$
10. $\sin(xy) + x = y$ at $(0,0)$
11. $e^{xy} + y = x^2$ at $(1,0)$
12. $x^2 + y^2 + e^{xy} = 2$ at $(1,0)$

Find the equation of the tangent line at the given point.

13. $x^2 + y^2 = 25$ at $(3,4)$
14. $y^3 + x^3 = 2$ at $(1,1)$
15. $x^2y + y = x$ at $(0,0)$

3.9 - Related Rates

In related rates problems, two or more quantities are changing with respect to time. The goal is to find a relationship between the variables, differentiate with respect to t, and connect the given rate(s) to the unknown rate.

Step-by-step process:

1. Draw a clear diagram and label all changing quantities with variables. Write down the given information and what you want.
2. Write an equation relating those variables (geometry, trig, physics).
3. Differentiate implicitly with respect to t.
4. Substitute all known values at the instant of interest and solve for the desired rate.
   - Always include units (e.g., cm^2/s, ft/s, rad/s)
   - Include correct signs (negative rates for decreasing quantities.)

1. Circles and Spheres

Problems with circles and spheres track how a changing radius $r(t)$ is related to changes in geometric quantities like area A, circumference C, volume V, or surface area S.

$$A = \pi r^2, \quad C = 2\pi r, \quad V = \tfrac{4}{3}\pi r^3, \quad S = 4\pi r^2$$

Examples

1. A spherical balloon is being inflated at 60 cubic centimeters per second. When the radius is 5 centimeters, how fast is the radius increasing, and how fast is the surface area increasing?
2. A melting snowball's surface area is decreasing at 20 square centimeters per minute. When the radius is 4 centimeters, how fast is the radius changing? Then, at that same moment, how fast is the volume changing?
3. A circular ripple on a pond is expanding so that its circumference is increasing at 0.5 meters per second. When the radius is 6 meters, how fast is the area increasing?

2. Rectangles and Boxes

Problems with rectangles and boxes relate changing sides to area, perimeter, volume, or surface area.

$$A_{\text{rect}} = xy, \quad P_{\text{rect}} = 2(x+y), \qquad V_{\text{cube}} = s^3, \quad S_{\text{cube}} = 6s^2$$

Examples

1. A rectangle's length is 10 cm and increasing at 2 cm/s; its width is 6 cm and increasing at 3 cm/s. At that instant, how fast are the area and perimeter increasing?

2. A rectangular garden has a constant area of 24 m^2. The length is increasing at 0.6 m/s. When the length is 6 m, how fast is the width changing? State whether the width is increasing or decreasing.

3. A cube's volume is increasing at 300 cubic centimeters per second. When the edge length is 10 cm, how fast are the edge length and the surface area increasing?

3. Cones and Cylinders

Problems with cones and cylinders relate how changing dimensions (radius $r(t)$ and/or height $h(t)$) drive changes in volume. For cones with a fixed shape, r is proportional to h; use similar triangles to eliminate one variable before differentiating with respect to time.

$$V_{\text{cone}} = \tfrac{1}{3}\pi r^2 h \qquad\qquad V_{\text{cyl}} = \pi r^2 h$$

Examples

1. A right circular cone points down with height 6 m and top radius 3 m. Water is pumped in at two cubic meters per minute. How fast is the water depth rising when the depth is 2 m?

2. Dry sand pours onto the ground at one and a half cubic meters per minute, forming a right circular cone whose radius is always half of its height. How fast is the height of the pile increasing when the pile is 0.8 m tall? Also report the rate at which the radius is changing at that instant.

3. A vertical cylindrical tank of constant radius 1.5 m has an open top; liquid drains so that the volume inside decreases at two tenths of a cubic meter per minute. Assuming the radius of the tank is fixed, how fast is the fluid level (height) falling when the depth is 2 m?

4. Right Triangles: Ladders, Motion, and Shadows

Problems in this family use right–triangles: relate legs and hypotenuse with the Pythagorean Theorem for ladders or perpendicular motion, and use similar triangles for shadow/spotlight setups.

Ladder (fixed length): $x^2 + y^2 = L^2$ (L constant)
Perpendicular motion: $x^2 + y^2 = d^2$ (d is separation)
Streetlamp/person (similar triangles): $\dfrac{H}{x+s} = \dfrac{h}{s} \iff Hs = h(x+s)$

Examples

1. A 13-ft ladder leans against a wall. The base slides away from the wall at *2 ft/s*. How fast is the top sliding down when the base is *5 ft* from the wall?

2. At a certain instant, Car A is *3 miles east* of an intersection and moving east at *40 mph*, while Car B is *4 miles north* and moving north at *30 mph*. At that instant, how fast is the distance between the cars changing?

3. A *6-ft-tall* person walks away from a *15-ft* streetlamp at *3 ft/s*. How fast is the tip of the shadow moving when the person is *20 ft* from the lamp?

5. Rotating Angles

These problems track how an angle (angle of elevation, bearing, or a rotating beam) changes as an object moves. Relate the angle $\theta(t)$ to a right triangle using $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.

$$\tan\theta \;=\; \frac{y}{x}$$

Examples

1. An observer on level ground watches a plane flying in a straight line at a constant altitude of 5,000 ft. The plane is moving *away* from the observer with horizontal speed 400 ft/s. How fast is the angle of elevation changing when the plane is 12,000 ft horizontally from the observer?

2. A lighthouse stands *2 miles* off a straight shoreline. A ship sails parallel to the shore at *12 mph*, away from the lighthouse. How fast is the bearing angle from the lighthouse to the ship changing when the ship is *4 miles* down the coast from the point closest to the lighthouse?

3. A searchlight located *0.5 miles* from a straight wall rotates at a constant 0.30 *radians per second*. How fast is the light spot moving along the wall when the beam makes a 60° angle with the perpendicular to the wall?

6. Mixed Motion (Non-Perpendicular Paths)

When two objects move on paths meeting at a changing angle $\phi(t)$ (or not at 90°), relate the separation $d(t)$ to path lengths $a(t)$, $b(t)$ via the Law of Cosines, then differentiate.

$$d^2 \;=\; a^2 + b^2 - 2ab\cos\phi$$

Examples

1. Two hikers leave the same point; Hiker A walks straight north at *4 km/h*. Hiker B walks at *5 km/h* on a path turning so that the angle between their paths increases at *2°/min*. How fast is their distance changing after *15 minutes*?

7. Motion Constrained to a Curve

A point moves along a curve. Treat $y = y(x(t))$ and use implicit differentiation to link dx/dt and dy/dt.

Examples

1. A point moves on $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. When it is at $\left(2, \frac{2\sqrt{5}}{3}\right)$, the x-coordinate increases at 0.3 units/s. How fast is y changing?

2. A bead slides on the curve $x^{2/3} + y^{2/3} = 4^{2/3}$. At the point $(\sqrt{2}, \sqrt{2})$, the horizontal speed is 1 cm/s to the right. Find the vertical speed (state up or down).