

Midterm 2 Study Guide – Solutions

MATH1300 - Calculus I

Fall 2025

Contents

2.8 - The Derivative Function	1
4.3 - What f' and f'' say about f	2
3.1 - Derivatives of Polynomials and Exponential Functions	3
3.2 - The Product and Quotient Rules	4
3.3 - Derivatives of Trig Functions	6
3.4 - The Chain Rule	9
3.5 - Implicit Differentiation	16
3.9 - Related Rates	19

2.8 - The Derivative Function

1. This is the difference quotient for $f(x) = \ln x$.

2.

$$g'(x) = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h}.$$

3. This is the derivative of $\sin x$ at $x = \frac{\pi}{6}$.

$$\lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - \sin\left(\frac{\pi}{6}\right)}{h} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

4. This is the derivative of $F(x) = x^5 - 4x$ at $x = 2$:

$$\lim_{x \rightarrow 2} \frac{x^5 - 4x - (2^5 - 4 \cdot 2)}{x - 2} = \lim_{x \rightarrow 2} \frac{(x^5 - 4x) - (32 - 8)}{x - 2}$$

$$F'(x) = 5x^4 - 4 \Rightarrow F'(2) = 5 \cdot 2^4 - 4 = 80 - 4 = 76.$$

5.

$$f(x) = \begin{cases} ax^2 + b, & x < 2, \\ 3x + 1, & x \geq 2. \end{cases}$$

Continuity at 2: $4a + b = 3 \cdot 2 + 1 = 7$.

One-sided derivatives:

$$f'_-(2) = 4a, \quad f'_+(2) = 3.$$

Differentiability: $4a = 3 \Rightarrow a = \frac{3}{4}$. Then $4(\frac{3}{4}) + b = 7 \Rightarrow b = 4$.

$$\text{Answer: } a = \frac{3}{4}, \quad b = 4.$$

6.

(a) $g(x) = |x|^{2/3}$. Use the definition:

$$g'(0) = \lim_{h \rightarrow 0} \frac{|h|^{2/3} - 0}{h},$$

which diverges to $+\infty$ from the right and $-\infty$ from the left. *Not differentiable* at 0.

(b) $h(x) = x|x|$. Piecewise: $h(x) = x^2$ for $x \geq 0$, and $h(x) = -x^2$ for $x < 0$. It follows that h is differentiable at 0 with $h'(0) = 0$.

7.

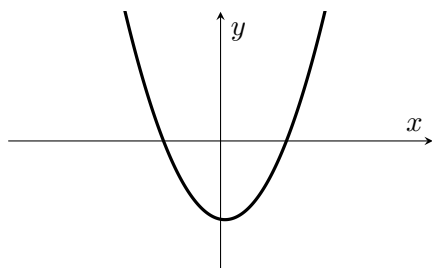
(a) Yes, f is continuous at $x = 1$.

(b) Derivatives: for $x < 1$, $f'(x) = 2x$; for $x > 1$, $f'(x) = 1$.

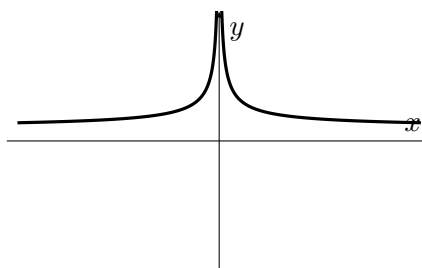
(c) $f'_-(1) = 2$, $f'_+(1) = 1$. Since $2 \neq 1$, f is *not differentiable* at $x = 1$.

4.3 - What f' and f'' say about f

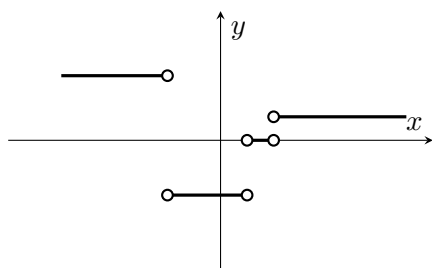
1. For each of the following graphs of $f(x)$, sketch the graph of $f'(x)$.



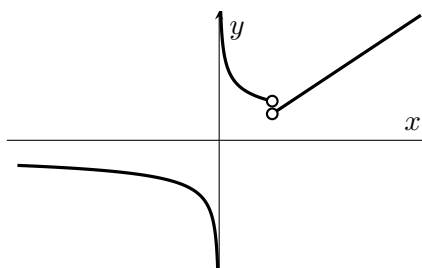
(i)



(ii)



(iii)



(iv)

- | | |
|---|---|
| <p>2. (i)</p> <ul style="list-style-type: none"> • increasing: $(-2, 0.25) \cup (2, \infty)$. • decreasing: $(-\infty, -2) \cup (0.25, 2)$. • concave up: $(-\infty, -1.1) \cup (1.25, \infty)$ • concave down: $(-1.1, 1.25)$. <p>(iii)</p> <ul style="list-style-type: none"> • increasing: $(-\infty, -1.3) \cup (3, \infty)$. • decreasing: $(-1.3, 3)$. • concave up: $(2.4, \infty)$. • concave down: $(-\infty, -1) \cup (0.8, 1.6)$. <p>(v)</p> <ul style="list-style-type: none"> • increasing: $(0, \infty)$. • decreasing: $(-\infty, 0)$. • concave up: $(-1.8, 1.8)$. • concave down: $(-\infty, -1.8) \cup (1.8, \infty)$ | <p>(ii)</p> <ul style="list-style-type: none"> • increasing: $(-\infty, -3.2) \cup (0, 3.2)$. • decreasing: $(-3.2, 0) \cup (3.2, \infty)$. • concave up: $(-1.6, 1.6)$. • concave down: $(-\infty, -1.6) \cup (1.6, \infty)$ <p>(iv)</p> <ul style="list-style-type: none"> • increasing: $(-\infty, 0)$. • decreasing: $(0, \infty)$. • concave up: $(-\infty, -1.5) \cup (1.4, \infty)$. • concave down: $(-1.5, 1.4)$. <p>(vi)</p> <ul style="list-style-type: none"> • increasing: $(-0.4, 2.3)$. • decreasing: $(-3.5, -0.4) \cup (2.3, 3.5)$. • concave up: $(-1.8, 0.8) \cup (3, 3.5)$. • concave down: $(-3.5, -1.8) \cup (0.8, 3)$. |
|---|---|

3.1 - Derivatives of Polynomials and Exponential Functions

1. $f(x) = x^{54} - 2x + 10$

$$f'(x) = 54x^{53} - 2.$$

2. $g(x) = \frac{1}{5}x^3 - 9x + 20$

$$g'(x) = \frac{3}{5}x^2 - 9.$$

3. $f(t) = 12e^t$

$$\frac{df}{dt} = 12e^t.$$

4. $F(t) = t^6 - e^4$ (e^4 is a constant)

$$F'(t) = 6t^5.$$

5. $r(z) = z^{-3/2} + z^{-5}$

$$r'(z) = -\frac{3}{2}z^{-5/2} - 5z^{-6}.$$

6. $g(x) = \frac{2}{\sqrt{x}} - \frac{3}{x^4} = 2x^{-1/2} - 3x^{-4}$

$$g'(x) = -x^{-3/2} + 12x^{-5} = -\frac{1}{x^{3/2}} + \frac{12}{x^5}.$$

7. $f(x) = x^2(x - 5)^2 = x^2(x^2 - 10x + 25) = x^4 - 10x^3 + 25x^2$

$$f'(x) = 4x^3 - 30x^2 + 50x.$$

8. $y = \frac{7 + 4x}{2x^3} = \frac{7}{2}x^{-3} + 2x^{-2}$

$$y' = -\frac{21}{2}x^{-4} - 4x^{-3} = -\frac{21}{2x^4} - \frac{4}{x^3} = -\frac{21 + 8x}{2x^4}.$$

3.2 - The Product and Quotient Rules

1. Differentiate $y = (2x^2 - 5)(x^3 + 4x)$.

$$\begin{aligned}y' &= (2x^2 - 5)'(x^3 + 4x) + (2x^2 - 5)(x^3 + 4x)' \\&= (4x)(x^3 + 4x) + (2x^2 - 5)(3x^2 + 4) \\&= 4x^4 + 16x^2 + (6x^4 - 7x^2 - 20) \\&= 10x^4 + 9x^2 - 20.\end{aligned}$$

2. Differentiate $f(x) = x^2 e^{2x}$.

$$f' = (x^2)' e^{2x} + x^2 (e^{2x})' = 2x e^{2x} + x^2 \cdot 2e^{2x} = e^{2x}(2x + 2x^2) = 2xe^{2x}(1 + x).$$

3. Differentiate $g(x) = \frac{5x}{e^x} = 5x e^{-x}$.

$$g' = 5e^{-x} + 5x(-e^{-x}) = 5e^{-x}(1 - x) = \frac{5(1 - x)}{e^x}.$$

4. Differentiate $h(t) = \frac{t^2 - 3}{t^3 + 1}$.

$$h' = \frac{(2t)(t^3 + 1) - (t^2 - 3)(3t^2)}{(t^3 + 1)^2} = \frac{2t^4 + 2t - 3t^4 + 9t^2}{(t^3 + 1)^2} = \frac{-t^4 + 9t^2 + 2t}{(t^3 + 1)^2}.$$

5. Differentiate $y = \frac{1 + x}{3 + e^x}$.

$$y' = \frac{(1)(3 + e^x) - (1 + x)(e^x)}{(3 + e^x)^2} = \frac{3 + e^x - e^x - xe^x}{(3 + e^x)^2} = \frac{3 - xe^x}{(3 + e^x)^2}.$$

6. Differentiate $y = \frac{x^2 e^x}{1 + x}$.

$$\begin{aligned}y' &= \frac{(x^2 e^x)'(1 + x) - (x^2 e^x)(1)}{(1 + x)^2} = \frac{e^x(2x + x^2)(1 + x) - x^2 e^x}{(1 + x)^2} \\&= \frac{e^x(2x + 3x^2 + x^3 - x^2)}{(1 + x)^2} = \frac{e^x x(x^2 + 2x + 2)}{(1 + x)^2}.\end{aligned}$$

7. Tangent line to $y = \frac{x^2}{4 + x^2}$ at $(2, \frac{1}{2})$.

$$y' = \frac{(4 + x^2)(2x) - (x^2)(2x)}{(4 + x^2)^2} = \frac{8x}{(4 + x^2)^2}.$$

At $x = 2$: $m = y'(2) = \frac{8 \cdot 2}{(8)^2} = \frac{1}{4}$. Point-slope through $(2, \frac{1}{2})$:

$$y - \frac{1}{2} = \frac{1}{4}(x - 2) \implies y = \frac{1}{4}x.$$

8. Given $f(2) = 3$, $f'(2) = -1$, $g(2) = -4$, $g'(2) = 5$. Compute:

$$(fg)'(2), \quad \left(\frac{f}{g}\right)'(2), \quad \left(\frac{g}{f}\right)'(2).$$

$$(fg)'(2) = f'(2)g(2) + f(2)g'(2) = (-1)(-4) + (3)(5) = 4 + 15 = 19,$$

$$\left(\frac{f}{g}\right)'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(-1)(-4) - (3)(5)}{(-4)^2} = \frac{4 - 15}{16} = -\frac{11}{16},$$

$$\left(\frac{g}{f}\right)'(2) = \frac{g'(2)f(2) - g(2)f'(2)}{[f(2)]^2} = \frac{(5)(3) - (-4)(-1)}{3^2} = \frac{15 - 4}{9} = \frac{11}{9}.$$

3.3 - Derivatives of Trig Functions

1. Differentiate $p(t) = 4 \cos t - 3 \sin t$.

$$p'(t) = -4 \sin t - 3 \cos t.$$

2. Differentiate $y = \sec x - 2 \tan x$.

$$\frac{dy}{dx} = \sec x \tan x - 2 \sec^2 x.$$

3. Evaluate $\frac{d}{du}(u^3 + \cot u)$.

$$\frac{d}{du}(u^3 + \cot u) = 3u^2 - \csc^2 u.$$

4. Differentiate $h(\theta) = \theta^2 \sin \theta \cos \theta$.

$$\begin{aligned} h'(\theta) &= \theta^2 \cdot \frac{d}{d\theta}(\sin \theta \cos \theta) + \sin \theta \cos \theta \cdot \frac{d}{d\theta}(\theta^2) \\ &= \theta^2 \cdot \left[\sin \theta \cdot \frac{d}{d\theta}(\cos \theta) + \cos \theta \cdot \frac{d}{d\theta}(\sin \theta) \right] + 2\theta \sin \theta \cos \theta \\ &= \theta^2 (\cos^2 \theta - \sin^2 \theta) + 2\theta \sin \theta \cos \theta \\ &= \theta^2 \cos(2\theta) + \theta \sin(2\theta) \end{aligned}$$

5. Differentiate $y = \cot z + \csc z$.

$$\frac{dy}{dz} = -\csc^2 z - \csc z \cot z.$$

6. Find the 205th derivative of $\sin x$.

Differentiate a few times to see the repeating pattern:

$$\begin{aligned} \frac{d^0}{dx^0} \sin x &= \sin x, \\ \frac{d^1}{dx^1} \sin x &= \cos x, \\ \frac{d^2}{dx^2} \sin x &= -\sin x, \\ \frac{d^3}{dx^3} \sin x &= -\cos x, \\ \frac{d^4}{dx^4} \sin x &= \sin x \quad (\text{cycle repeats every 4}). \end{aligned}$$

So the n -th derivative depends on the remainder n when we divide by 4:

$$n \equiv 0 : \sin x, \quad n \equiv 1 : \cos x, \quad n \equiv 2 : -\sin x, \quad n \equiv 3 : -\cos x.$$

Since $205 = 4 \cdot 51 + 1$, we have $205 \equiv 1 \pmod{4}$. Therefore

$$\frac{d^{205}}{dx^{205}} \sin x = \cos x.$$

7. Find the 1900th derivative of $\cos x$.

Again, observe the 4-term cycle:

$$\begin{aligned}\frac{d^0}{dx^0} \cos x &= \cos x, \\ \frac{d^1}{dx^1} \cos x &= -\sin x, \\ \frac{d^2}{dx^2} \cos x &= -\cos x, \\ \frac{d^3}{dx^3} \cos x &= \sin x, \\ \frac{d^4}{dx^4} \cos x &= \cos x \quad (\text{cycle repeats every 4}).\end{aligned}$$

So the n -th derivative depends on the remainder n when we divide by 4:

$$n \equiv 0 : \cos x, \quad n \equiv 1 : -\sin x, \quad n \equiv 2 : -\cos x, \quad n \equiv 3 : \sin x.$$

Since $1900 = 4 \cdot 475 + 0$, we have $1900 \equiv 0 \pmod{4}$. Therefore

$$\frac{d^{1900}}{dx^{1900}} \cos x = \cos x.$$

8. Compute $\frac{d}{dx}(e^x \csc x)$.

$$\frac{d}{dx}(e^x \csc x) = e^x \csc x + e^x(-\csc x \cot x) = e^x \csc x (1 - \cot x).$$

9. Compute $\frac{d}{ds}(\sec s \tan s)$.

$$\frac{d}{ds}(\sec s \tan s) = \sec s \tan^2 s + \sec s \sec^2 s = \sec s (\tan^2 s + \sec^2 s) = \sec s (2 \sec^2 s - 1).$$

10. Find $f'(x)$ if $f(x) = \frac{\tan x}{1 + \tan^2 x}$.

$$f(x) = \frac{\tan x}{\sec^2 x} = \sin x \cos x \Rightarrow f'(x) = \cos^2 x - \sin^2 x = \cos(2x).$$

11. Find $g'(x)$ if $g(x) = \frac{\cos x}{1 - \sin x}$.

Let $u = \cos x$ and $v = 1 - \sin x$. Then $u' = -\sin x$ and $v' = -\cos x$. Hence

$$g'(x) = \frac{u'v - uv'}{v^2} = \frac{(-\sin x)(1 - \sin x) - \cos x(-\cos x)}{(1 - \sin x)^2}.$$

Simplify the numerator:

$$(-\sin x)(1 - \sin x) + \cos^2 x = -\sin x + \sin^2 x + \cos^2 x = 1 - \sin x.$$

Therefore

$$g'(x) = \frac{1 - \sin x}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}.$$

(Equivalent form.) Using $\cos^2 x = 1 - \sin^2 x$,

$$\frac{1}{1 - \sin x} = \frac{1 + \sin x}{\cos^2 x} = \sec x \tan x + \sec^2 x.$$

Both forms are correct.

12. Tangent to $y = e^x \sin x + \cos x$ at $(0, 1)$.

Slope:

$$y'(x) = \frac{d}{dx}(e^x \sin x) + \frac{d}{dx}(\cos x) = e^x(\sin x + \cos x) - \sin x.$$

At $x = 0$: $m = y'(0) = 1$. Point-slope:

$$y - 1 = 1(x - 0) \implies y = x + 1.$$

13. Tangent to $y = \csc x$ at $x = \frac{\pi}{3}$.

Value: $y(\pi/3) = \csc(\pi/3) = \frac{2}{\sqrt{3}}$. Slope: $y' = -\csc x \cot x$, so

$$m = y'(\frac{\pi}{3}) = -\frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = -\frac{2}{3}.$$

Line through $(\frac{\pi}{3}, \frac{2}{\sqrt{3}})$ with slope $-\frac{2}{3}$:

$$y - \frac{2}{\sqrt{3}} = -\frac{2}{3} \left(x - \frac{\pi}{3} \right).$$

14. Find all x in $[0, 2\pi]$ where the tangent to $g(x) = \sin x + \cos x$ is parallel to the line $y = -x$.

Parallel to $y = -x$ means slope -1 . Compute

$$g'(x) = \cos x - \sin x = \sqrt{2} \cos\left(x + \frac{\pi}{4}\right).$$

Set $g'(x) = -1$:

$$\sqrt{2} \cos\left(x + \frac{\pi}{4}\right) = -1 \implies \cos\left(x + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Thus

$$x + \frac{\pi}{4} = \frac{3\pi}{4} \quad \text{or} \quad x + \frac{\pi}{4} = \frac{5\pi}{4}$$

So the tangent is parallel to $y = -x$ at $x = \frac{\pi}{2}$, π on $[0, 2\pi]$.

15. Where does the tangent to $y = \tan x$ have slope 3 on $(-\frac{\pi}{2}, \frac{\pi}{2})$?

Since $y' = \sec^2 x$, solve $\sec^2 x = 3$. Equivalently, $1 + \tan^2 x = 3 \Rightarrow \tan^2 x = 2 \Rightarrow \tan x = \pm\sqrt{2}$. On $(-\frac{\pi}{2}, \frac{\pi}{2})$ this gives

$$x = \arctan(\sqrt{2}) \quad \text{and} \quad x = -\arctan(\sqrt{2}) \quad (\text{equivalently, } x = \pm \arccos(1/\sqrt{3})).$$

3.4 - The Chain Rule

1. $y = \sin((3x - 1)^4)$

(a) *Leibniz notation:*

$$y = \sin v, \quad v = w^4, \quad w = 3x - 1.$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}.$$

Compute the pieces:

$$\frac{dy}{dv} = \cos v, \quad \frac{dv}{dw} = 4w^3, \quad \frac{dw}{dx} = 3.$$

Assemble and substitute $v = w^4$, $w = 3x - 1$:

$$\frac{dy}{dx} = \cos(v) \cdot 4w^3 \cdot 3 = 12(3x - 1)^3 \cos((3x - 1)^4).$$

(b) *Function notation:*

$$y = f(g(h(x))), \quad f(u) = \sin u, \quad g(t) = t^4, \quad h(x) = 3x - 1.$$

Chain rule:

$$(f \circ g \circ h)'(x) = f'(g(h(x))) g'(h(x)) h'(x).$$

With $f'(u) = \cos u$, $g'(t) = 4t^3$, $h'(x) = 3$:

$$\frac{dy}{dx} = \cos((3x - 1)^4) \cdot 4(3x - 1)^3 \cdot 3 = 12(3x - 1)^3 \cos((3x - 1)^4).$$

2. $y = (\cos(2x - 1))^5$

(a) *Leibniz notation:*

$$y = v^5, \quad v = \cos w, \quad w = 2x - 1.$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}.$$

Compute the pieces:

$$\frac{dy}{dv} = 5v^4, \quad \frac{dv}{dw} = -\sin w, \quad \frac{dw}{dx} = 2.$$

Assemble and substitute $v = \cos w$, $w = 2x - 1$:

$$\frac{dy}{dx} = (5v^4)(-\sin w)(2) = -10(\cos w)^4 \sin w = -10 \sin(2x - 1) \cos^4(2x - 1).$$

(b) *Function notation:*

$$y = f(g(h(x))), \quad f(u) = u^5, \quad g(t) = \cos t, \quad h(x) = 2x - 1.$$

Chain rule:

$$(f \circ g \circ h)'(x) = f'(g(h(x))) g'(h(x)) h'(x).$$

With $f'(u) = 5u^4$, $g'(t) = -\sin t$, $h'(x) = 2$:

$$\frac{dy}{dx} = 5(\cos(2x-1))^4 (-\sin(2x-1)) (2) = -10 \sin(2x-1) \cos^4(2x-1).$$

3. $y = \sqrt{1 + \cot(x^2 - x)}$

(a) *Leibniz notation:*

$$y = u^{1/2}, \quad u = 1 + \cot v, \quad v = x^2 - x.$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

Compute the pieces:

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2}, \quad \frac{du}{dv} = -\csc^2 v, \quad \frac{dv}{dx} = 2x - 1.$$

Assemble and substitute $u = 1 + \cot(x^2 - x)$, $v = x^2 - x$:

$$\frac{dy}{dx} = -\frac{(2x-1) \csc^2(x^2-x)}{2\sqrt{1+\cot(x^2-x)}}.$$

(b) *Function notation:*

$$y = f(g(h(x))), \quad f(u) = u^{1/2}, \quad g(v) = 1 + \cot v, \quad h(x) = x^2 - x.$$

Chain rule:

$$(f \circ g \circ h)'(x) = f'(g(h(x))) g'(h(x)) h'(x),$$

which gives the same derivative as above.

4. $y = 3^{2x - \sin x}$

(a) *Leibniz notation:*

$$y = 3^u, \quad u = 2x - \sin x.$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \quad \frac{dy}{du} = (\ln 3) 3^u, \quad \frac{du}{dx} = 2 - \cos x.$$

Substitute $u = 2x - \sin x$:

$$\frac{dy}{dx} = 3^{2x - \sin x} \ln 3 (2 - \cos x).$$

(b) *Function notation:*

$$y = f(g(x)), \quad f(u) = 3^u, \quad g(x) = 2x - \sin x.$$

Chain rule:

$$(f \circ g)'(x) = f'(g(x)) g'(x) = \ln 3 3^{2x - \sin x} (2 - \cos x).$$

5. $y = (1 + e^{x^3})^7$

(a) *Leibniz notation:*

$$y = a^7, \quad a = 1 + b, \quad b = e^c, \quad c = x^3.$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{da} \cdot \frac{da}{db} \cdot \frac{db}{dc} \cdot \frac{dc}{dx}.$$

Pieces:

$$\frac{dy}{da} = 7a^6, \quad \frac{da}{db} = 1, \quad \frac{db}{dc} = e^c, \quad \frac{dc}{dx} = 3x^2.$$

Assemble and substitute $a = 1 + e^{x^3}$, $c = x^3$:

$$\frac{dy}{dx} = 7(1 + e^{x^3})^6 \cdot e^{x^3} \cdot 3x^2 = 21x^2 e^{x^3} (1 + e^{x^3})^6.$$

(b) *Function notation:*

$$y = f(g(h(k(x))))), \quad f(u) = u^7, \quad g(v) = 1 + v, \quad h(w) = e^w, \quad k(x) = x^3.$$

Chain rule:

$$(f \circ g \circ h \circ k)' = f'(g(h(k))) g'(h(k)) h'(k) k'(x),$$

which yields the same expression $21x^2 e^{x^3} (1 + e^{x^3})^6$.

6. $y = e^{\tan(1/\sqrt{x})} \quad (x > 0)$

(a) *Leibniz notation:*

$$y = e^u, \quad u = \tan v, \quad v = x^{-1/2}.$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

Pieces:

$$\frac{dy}{du} = e^u, \quad \frac{du}{dv} = \sec^2 v, \quad \frac{dv}{dx} = -\frac{1}{2}x^{-3/2}.$$

Assemble with $u = \tan(1/\sqrt{x})$, $v = 1/\sqrt{x}$:

$$\frac{dy}{dx} = -\frac{1}{2}x^{-3/2} e^{\tan(1/\sqrt{x})} \sec^2(1/\sqrt{x}).$$

(b) *Function notation:*

$$y = f(g(h(x))), \quad f(u) = e^u, \quad g(v) = \tan v, \quad h(x) = x^{-1/2}.$$

Chain rule:

$$(f \circ g \circ h)' = f'(g(h)) g'(h) h',$$

giving the same derivative.

7. $y = \sqrt{1 - \tan(2x)}$

(a) *Leibniz notation:*

$$y = m^{1/2}, \quad m = 1 - q, \quad q = \tan p, \quad p = 2x.$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dm} \cdot \frac{dm}{dq} \cdot \frac{dq}{dp} \cdot \frac{dp}{dx}.$$

Pieces:

$$\frac{dy}{dm} = \frac{1}{2}m^{-1/2}, \quad \frac{dm}{dq} = -1, \quad \frac{dq}{dp} = \sec^2 p, \quad \frac{dp}{dx} = 2.$$

Assemble and substitute $m = 1 - \tan(2x)$, $p = 2x$:

$$\frac{dy}{dx} = -\frac{\sec^2(2x)}{\sqrt{1 - \tan(2x)}}.$$

(b) *Function notation:*

$$y = f(g(h(k(x)))) , \quad f(u) = u^{1/2}, \quad g(v) = 1 - v, \quad h(w) = \tan w, \quad k(x) = 2x,$$

so

$$(f \circ g \circ h \circ k)' = f'(g(h(k))) g'(h(k)) h'(k) k'(x)$$

reproduces the same result.

8. $y = x^2 5^{\cos x}$

(a) *Leibniz notation:*

$$y = f \cdot g, \quad f = x^2, \quad g = 5^u, \quad u = \cos x.$$

Product rule + chain on g :

$$\frac{dy}{dx} = f' g + f g', \quad g' = \frac{dg}{du} \frac{du}{dx} = (\ln 5) 5^u (-\sin x).$$

Compute and substitute $u = \cos x$:

$$\frac{dy}{dx} = 2x 5^{\cos x} + x^2 5^{\cos x} \ln 5 (-\sin x) = 5^{\cos x} (2x - x^2 \ln 5 \sin x).$$

(b) *Function notation:*

$$y = p(x) q(x), \quad p(x) = x^2, \quad q(x) = r(s(x)), \quad r(a) = 5^a, \quad s(x) = \cos x.$$

Then

$$y' = p' q + p q', \quad q' = r'(s) s' = \ln 5 5^{\cos x} (-\sin x),$$

which gives the same simplified expression.

9. $y = \frac{7^{\tan x}}{\sin x} = 7^{\tan x} \csc x$

(a) *Leibniz notation:*

$$y = F \cdot G, \quad F = 7^u, \quad u = \tan x, \quad G = \csc x.$$

Product & chain rules:

$$\frac{dy}{dx} = F'G + FG', \quad F' = \frac{dF}{du} \frac{du}{dx}, \quad G' = \frac{dG}{dx}.$$

Pieces:

$$\frac{dF}{du} = (\ln 7) 7^u, \quad \frac{du}{dx} = \sec^2 x \implies F' = (\ln 7) 7^u \sec^2 x, \quad G' = -\csc x \cot x.$$

Assemble and substitute $u = \tan x$:

$$\frac{dy}{dx} = 7^{\tan x} \csc x (\ln 7 \sec^2 x - \cot x).$$

(b) *Function notation:*

$$y = p(x) q(x), \quad p(x) = t(s(x)), \quad t(a) = 7^a, \quad s(x) = \tan x, \quad q(x) = \csc x.$$

Product rule with chain on p :

$$(pq)' = p'q + pq', \quad p' = t'(s) s', \quad q' = -\csc x \cot x.$$

Here $t'(a) = (\ln 7) 7^a$ and $s'(x) = \sec^2 x$, so

$$p' = (\ln 7) 7^{\tan x} \sec^2 x, \quad q' = -\csc x \cot x,$$

and therefore

$$\frac{dy}{dx} = 7^{\tan x} \csc x (\ln 7 \sec^2 x - \cot x).$$

$$10. \quad y = \frac{e^{x^2}}{(1+x^4)^{3/2}} = e^{x^2} (1+x^4)^{-3/2}$$

(a) *Leibniz notation:*

$$y = A \cdot B, \quad A = e^v, \quad v = x^2, \quad B = w^{-3/2}, \quad w = 1 + x^4.$$

Product rule and inner chain rules:

$$\frac{dy}{dx} = A'B + AB', \quad A' = \frac{dA}{dv} \frac{dv}{dx} = e^v \cdot (2x), \quad B' = \frac{dB}{dw} \frac{dw}{dx} = \left(-\frac{3}{2} w^{-5/2}\right) \cdot (4x^3).$$

Assemble and substitute $v = x^2$, $w = 1 + x^4$:

$$\frac{dy}{dx} = e^{x^2} (2x) (1+x^4)^{-3/2} - 6x^3 e^{x^2} (1+x^4)^{-5/2}.$$

Factor to simplify:

$$\frac{dy}{dx} = \frac{2x e^{x^2} (1 - 3x^2 + x^4)}{(1+x^4)^{5/2}}.$$

(b) *Function notation:*

$$y = p(x)q(x), \quad p(x) = f(g(x)), \quad f(u) = e^u, \quad g(x) = x^2, \quad q(x) = r(s(x)), \quad r(a) = a^{-3/2}, \quad s(x) = 1+x^4.$$

Product rule with composition rules:

$$y' = p'q + pq', \quad p' = f'(g)g' = e^{x^2}(2x), \quad q' = r'(s)s' = \left(-\frac{3}{2}(1+x^4)^{-5/2}\right) \cdot (4x^3).$$

Thus

$$y' = e^{x^2}(2x)(1+x^4)^{-3/2} - 6x^3e^{x^2}(1+x^4)^{-5/2} = \frac{2xe^{x^2}(1-3x^2+x^4)}{(1+x^4)^{5/2}}.$$

11.

$$f(1) = 2, \quad f'(1) = -3, \quad g(2) = -4, \quad g'(2) = 5, \quad h(-4) = 1, \quad h'(-4) = 7, \quad H(x) = h(g(f(x))).$$

Chain rule:

$$H'(x) = h'(g(f(x))) \cdot g'(f(x)) \cdot f'(x).$$

Evaluate at $x = 1$:

$$H'(1) = h'(g(f(1))) \cdot g'(f(1)) \cdot f'(1) = h'(g(2)) \cdot g'(2) \cdot (-3) = h'(-4) \cdot 5 \cdot (-3) = 7 \cdot 5 \cdot (-3) = \boxed{-105}.$$

12.

$$f(2) = 1, \quad f'(2) = -5, \quad g(1) = 3, \quad g'(1) = 4, \quad G(x) = g(f(x^3)).$$

Chain rule:

$$G'(x) = g'(f(x^3)) \cdot f'(x^3) \cdot 3x^2.$$

At $x = \sqrt[3]{2}$, we have $x^3 = 2$, so

$$G'(\sqrt[3]{2}) = g'(f(2)) \cdot f'(2) \cdot 3(\sqrt[3]{2})^2 = g'(1) \cdot (-5) \cdot 3\sqrt[3]{4} = \boxed{-60\sqrt[3]{4}}.$$

13. Tangent line to $y = \frac{\sqrt{1+\cos x}}{1+x^2}$ at $x = 0$.

Write $y(x) = (1+\cos x)^{1/2}(1+x^2)^{-1}$.

$$y(0) = \frac{\sqrt{1+\cos 0}}{1+0} = \sqrt{2}.$$

Differentiate:

$$y'(x) = \left(\frac{1}{2}(1+\cos x)^{-1/2}(-\sin x)\right)(1+x^2)^{-1} + (1+\cos x)^{1/2}\left(-(1+x^2)^{-2} \cdot 2x\right).$$

Evaluate at $x = 0$: $\sin 0 = 0$, so both terms vanish and $y'(0) = 0$. Hence the tangent line is

$$\boxed{y = \sqrt{2}}.$$

14. Find all x in $[0, 2\pi]$ where the tangent to $y = e^{-x} \sin x$ is horizontal.

A horizontal tangent means $y'(x) = 0$. Differentiate:

$$y'(x) = \frac{d}{dx}(e^{-x} \sin x) = (-e^{-x}) \sin x + e^{-x} \cos x = e^{-x}(\cos x - \sin x).$$

Since $e^{-x} > 0$ for all x , we need

$$\cos x - \sin x = 0 \quad \Longleftrightarrow \quad \cos x = \sin x \quad \Longleftrightarrow \quad \tan x = 1.$$

Thus $x = \frac{\pi}{4} + k\pi$. On $[0, 2\pi]$ this gives

$$x = \frac{\pi}{4}, \frac{5\pi}{4}.$$

3.5 - Implicit Differentiation

1. $x^2 + xy + y^2 = 7$

Differentiate implicitly:

$$2x + (xy)' + 2y y' = 0 \Rightarrow 2x + x y' + y + 2y y' = 0.$$

Solve for y' :

$$y' = -\frac{2x + y}{x + 2y}.$$

2. $x^3 + y^3 = 6xy$

Differentiate:

$$3x^2 + 3y^2 y' = 6y + 6x y'.$$

Collect y' terms and solve:

$$y'(3y^2 - 6x) = 6y - 3x^2 \Rightarrow y' = \frac{2y - x^2}{y^2 - 2x}.$$

3. $x^2y + xy^2 = 6$

Differentiate:

$$(x^2y)' + (xy^2)' = 0 \Rightarrow (2x)y + x^2y' + y^2 + 2xy y' = 0.$$

Solve:

$$y'(x^2 + 2xy) = -(2xy + y^2) \Rightarrow y' = -\frac{2xy + y^2}{x^2 + 2xy}.$$

4. $\sin(x + y) = xy$

Differentiate:

$$\cos(x + y)(1 + y') = x y' + y.$$

Solve:

$$y'(\cos(x + y) - x) = y - \cos(x + y) \Rightarrow y' = \frac{y - \cos(x + y)}{\cos(x + y) - x}.$$

5. $\cos(xy) + y = x$

Differentiate:

$$-\sin(xy)(xy)' + y' = 1 \Rightarrow -\sin(xy)(x y' + y) + y' = 1.$$

Solve:

$$y'(1 - x \sin(xy)) = 1 + y \sin(xy) \Rightarrow y' = \frac{1 + y \sin(xy)}{1 - x \sin(xy)}.$$

6. $e^{x+y} = x^2 - y$

Differentiate:

$$e^{x+y}(1 + y') = 2x - y'.$$

Solve:

$$y'(e^{x+y} + 1) = 2x - e^{x+y} \Rightarrow y' = \frac{2x - e^{x+y}}{e^{x+y} + 1}.$$

7. $x^2 + y^2 = \sin(xy)$

Differentiate:

$$2x + 2y y' = \cos(xy) (xy)' = \cos(xy) (x y' + y).$$

Solve:

$$y'(2y - x \cos(xy)) = \cos(xy) y - 2x \Rightarrow y' = \frac{\cos(xy) y - 2x}{2y - x \cos(xy)}.$$

8. $\tan y = x e^y$

Differentiate:

$$\sec^2 y y' = e^y + x e^y y'.$$

Solve:

$$y'(\sec^2 y - x e^y) = e^y \Rightarrow y' = \frac{e^y}{\sec^2 y - x e^y}.$$

(Equivalently, using $\sec^2 y = 1 + \tan^2 y = 1 + x^2 e^{2y}$: $y' = \frac{e^y}{1 + x^2 e^{2y} - x e^y}$.)

9. $x^2 y + y^2 = 4$ at $(0, 2)$

Differentiate:

$$(x^2 y)' + (y^2)' = 0 \Rightarrow 2x y + x^2 y' + 2y y' = 0.$$

Solve for y' :

$$y'(x^2 + 2y) = -2xy \Rightarrow y' = -\frac{2xy}{x^2 + 2y}.$$

At $(0, 2)$: $y' = 0$.

10. $\sin(xy) + x = y$ at $(0, 0)$

Differentiate:

$$\cos(xy) (x y' + y) + 1 = y'.$$

Solve:

$$y' - x \cos(xy) y' = \cos(xy) y + 1 \Rightarrow y' = \frac{\cos(xy) y + 1}{1 - x \cos(xy)}.$$

At $(0, 0)$: $y' = 1$.

11. $e^{xy} + y = x^2$ at $(1, 0)$

Differentiate:

$$e^{xy} (x y' + y) + y' = 2x.$$

Solve:

$$y'(e^{xy} x + 1) = 2x - e^{xy} y \Rightarrow y' = \frac{2x - e^{xy} y}{e^{xy} x + 1}.$$

At $(1, 0)$: $y' = 1$.

12. $x^2 + y^2 + e^{xy} = 2$ at $(1, 0)$

Differentiate:

$$2x + 2y y' + e^{xy} (x y' + y) = 0.$$

At $(1, 0)$: $2(1) + 2(0)y' + e^0(1 \cdot y' + 0) = 0 \Rightarrow 2 + y' = 0$. Thus $y' = -2$.

13. $x^2 + y^2 = 25$ at $(3, 4)$

Differentiate:

$$2x + 2y y' = 0 \implies y' = -\frac{x}{y}.$$

At $(3, 4)$: $m = -\frac{3}{4}$. Tangent line (point-slope):

$$y - 4 = -\frac{3}{4}(x - 3).$$

(Equivalently, $y = -\frac{3}{4}x + \frac{25}{4}$.)

14. $y^3 + x^3 = 2$ at $(1, 1)$

Differentiate:

$$3y^2 y' + 3x^2 = 0 \implies y' = -\frac{x^2}{y^2}.$$

At $(1, 1)$: $m = -1$. Tangent line:

$$y - 1 = -1(x - 1) \iff y = -x + 2.$$

15. $x^2 y + y = x$ at $(0, 0)$

Differentiate:

$$(x^2 y)' + y' = 1 \implies 2xy + x^2 y' + y' = 1.$$

Solve for y' :

$$y'(x^2 + 1) = 1 - 2xy \implies y' = \frac{1 - 2xy}{x^2 + 1}.$$

At $(0, 0)$: $m = 1$. Tangent line:

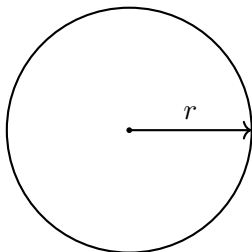
$$y - 0 = 1(x - 0) \iff y = x.$$

3.9 - Related Rates

1. Circles and Spheres

1. A spherical balloon is being inflated at 60 cubic centimeters per second. When the radius is 5 centimeters, how fast is the radius increasing, and how fast is the surface area increasing?

(a)



Given: $\frac{dV}{dt} = 60 \text{ cm}^3/\text{s}$

Want: $\frac{dr}{dt}$ and $\frac{dS}{dt}$ at $r = 5 \text{ cm}$.

(b) **Relationship.** Sphere formulas:

$$V = \frac{4}{3}\pi r^3, \quad S = 4\pi r^2.$$

(c) **Differentiate.** With respect to time t :

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}, \quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

(d) **Plug in & solve.** At $r = 5 \text{ cm}$ and $\frac{dV}{dt} = 60 \text{ cm}^3/\text{s}$,

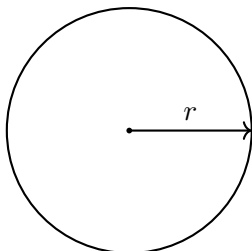
$$\frac{dr}{dt} = \frac{1}{4\pi r^2} \frac{dV}{dt} = \frac{60}{4\pi \cdot 25} = \frac{3}{5\pi} \text{ cm/s} \approx 0.191 \text{ cm/s}.$$

Then

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi \cdot 5 \cdot \frac{3}{5\pi} = 24 \text{ cm}^2/\text{s}.$$

2. A melting snowball's surface area is decreasing at 20 square centimeters per minute. When the radius is 4 centimeters, how fast is the radius changing? Then, at that same moment, how fast is the volume changing?

(a)



Given: $\frac{dS}{dt} = -20 \text{ cm}^2/\text{min}$

Want: $\frac{dr}{dt}$ and $\frac{dV}{dt}$ at $r = 4 \text{ cm}$.

(b) **Relationship.** Sphere formulas:

$$S = 4\pi r^2, \quad V = \frac{4}{3}\pi r^3.$$

(c) **Differentiate.** With respect to time t :

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}, \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

(d) **Plug in & solve.** From $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$,

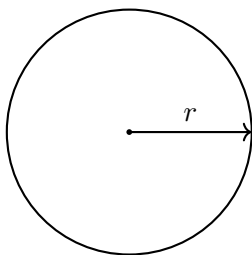
$$\frac{dr}{dt} = \frac{1}{8\pi r} \frac{dS}{dt} = \frac{-20}{8\pi \cdot 4} = -\frac{5}{8\pi} \text{ cm/min} \approx -0.199 \text{ cm/min}.$$

Then use $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ at $r = 4$:

$$\frac{dV}{dt} = 4\pi(4)^2 \left(-\frac{5}{8\pi}\right) = 64\pi \left(-\frac{5}{8\pi}\right) = -40 \text{ cm}^3/\text{min}.$$

3. A circular ripple on a pond is expanding so that its circumference is increasing at 0.5 meters per second. When the radius is 6 meters, how fast is the area increasing?

(a)



Given: $\frac{dC}{dt} = 0.5 \text{ m/s}$

Want: $\frac{dA}{dt}$ at $r = 6 \text{ m}$.

(b) Circle formulas:

$$C = 2\pi r, \quad A = \pi r^2.$$

(c) With respect to time t :

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}, \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

(d) From $\frac{dC}{dt} = 2\pi \frac{dr}{dt}$,

$$\frac{dr}{dt} = \frac{0.5}{2\pi} = \frac{1}{4\pi} \text{ m/s}.$$

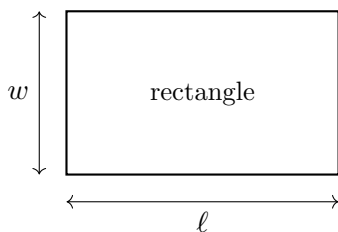
Then at $r = 6 \text{ m}$,

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(6) \left(\frac{1}{4\pi}\right) = 3 \text{ m}^2/\text{s}.$$

2. Rectangles and Boxes

1. A rectangle's length is 10 cm and increasing at 2 cm/s; its width is 6 cm and increasing at 3 cm/s. At that instant, how fast are the area and perimeter increasing?

(a)



Given: $\ell = 10 \text{ cm}$, $\frac{d\ell}{dt} = 2 \text{ cm/s}$; $w = 6 \text{ cm}$, $\frac{dw}{dt} = 3 \text{ cm/s}$.

Want: $\frac{dA}{dt}$ and $\frac{dP}{dt}$ at this instant.

(b) Relationships:

$$A = \ell w, \quad P = 2(\ell + w).$$

(c) Differentiate with respect to time t :

$$\frac{dA}{dt} = \ell \frac{dw}{dt} + w \frac{d\ell}{dt}, \quad \frac{dP}{dt} = 2 \left(\frac{d\ell}{dt} + \frac{dw}{dt} \right).$$

(d) Plug in and solve:

$$\frac{dA}{dt} = (10)(3) + (6)(2) = 30 + 12 = 42 \text{ cm}^2/\text{s}, \quad \frac{dP}{dt} = 2(2 + 3) = 10 \text{ cm/s}.$$

$$\frac{dA}{dt} = 42 \text{ cm}^2/\text{s}, \quad \frac{dP}{dt} = 10 \text{ cm/s}$$

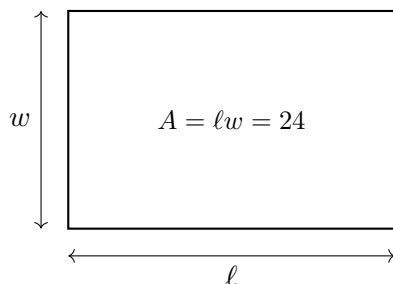
2. A rectangular garden has a constant area of 24 m^2 . The length is increasing at 0.6 m/s . When the length is 6 m , how fast is the width changing? State whether the width is increasing or decreasing.

(a)

$$\frac{d\ell}{dt} = +0.6 \text{ m/s}, \quad \frac{dw}{dt} = ?$$

Given: $A = 24 \text{ m}^2$, $\frac{d\ell}{dt} = 0.6 \text{ m/s}$, $\ell = 6 \text{ m}$.

Want: $\frac{dw}{dt}$ when $\ell = 6 \text{ m}$.



(b) Relationship: $A = \ell w = 24$ (constant).

(c) Differentiate with respect to time t :

$$\frac{dA}{dt} = \ell \frac{dw}{dt} + w \frac{d\ell}{dt} = 0 \quad \Rightarrow \quad \ell \frac{dw}{dt} = -w \frac{d\ell}{dt}.$$

(d) Compute w first: with $\ell w = 24$ and $\ell = 6$, we have $w = 24/6 = 4 \text{ m}$. Then

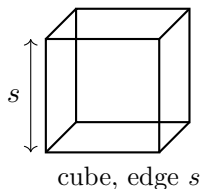
$$\frac{dw}{dt} = -\frac{w}{\ell} \frac{d\ell}{dt} = -\frac{4}{6} (0.6) = -0.4 \text{ m/s}.$$

$$\frac{dw}{dt} = -0.4 \text{ m/s}$$

The width is *decreasing*.

3. A cube's volume is increasing at 300 cubic centimeters per second. When the edge length is 10 cm, how fast are the edge length and the surface area increasing?

(a)



Given: $\frac{dV}{dt} = 300 \text{ cm}^3/\text{s}$, $s = 10 \text{ cm}$.

Want: $\frac{ds}{dt}$ and $\frac{dS}{dt}$ at $s = 10 \text{ cm}$.

(b) Relationships:

$$V = s^3, \quad S = 6s^2.$$

(c) Differentiate with respect to time t :

$$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}, \quad \frac{dS}{dt} = 12s \frac{ds}{dt}.$$

(d) Plug in and solve. From $\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$ with $s = 10$:

$$\frac{ds}{dt} = \frac{300}{3 \cdot 10^2} = \frac{300}{300} = 1 \text{ cm/s}.$$

Then

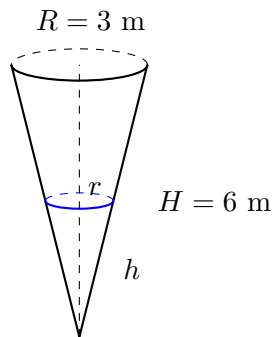
$$\frac{dS}{dt} = 12s \frac{ds}{dt} = 12(10)(1) = 120 \text{ cm}^2/\text{s}.$$

$\frac{ds}{dt} = 1 \text{ cm/s}, \quad \frac{dS}{dt} = 120 \text{ cm}^2/\text{s}$

3. Cones and Cylinders

1. A right circular cone points down with height 6 m and top radius 3 m. Water is pumped in at two cubic meters per minute. How fast is the water depth rising when the depth is 2 m?

(a)



Given: $H = 6 \text{ m}$, $R = 3 \text{ m}$, $\frac{dV}{dt} = +2 \text{ m}^3/\text{min}$, $h = 2 \text{ m}$.

Want: $\frac{dh}{dt}$ when $h = 2 \text{ m}$.

(b) *Relationship.* Similar triangles give $\frac{r}{h} = \frac{R}{H} = \frac{3}{6} = \frac{1}{2} \Rightarrow r = \frac{1}{2}h$.

Cone volume: $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3$.

(c) Differentiate with respect to time.

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

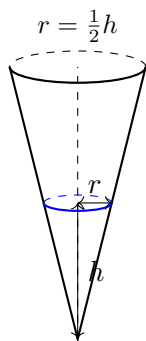
(d) Plug in and solve. At $h = 2$ m and $\frac{dV}{dt} = 2$ m³/min,

$$2 = \frac{\pi}{4} (2)^2 \frac{dh}{dt} = \pi \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{2}{\pi} \text{ m/min} \approx 0.637 \text{ m/min}.$$

$$\frac{dh}{dt} = \frac{2}{\pi} \text{ m/min}$$

2. Dry sand pours onto the ground at one and a half cubic meters per minute, forming a right circular cone whose radius is always half of its height. How fast is the height of the pile increasing when the pile is 0.8 m tall? Also report the rate at which the radius is changing at that instant.

(a)



right circular cone

Given: $\frac{dV}{dt} = 1.5$ m³/min, $r = \frac{1}{2}h$, $h = 0.8$ m.

Want: $\frac{dh}{dt}$ and $\frac{dr}{dt}$ at $h = 0.8$ m.

(b) Relationship (eliminate r using similar triangles). For a cone,

$$V = \frac{1}{3}\pi r^2 h, \quad r = \frac{1}{2}h \implies V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3.$$

(c) Differentiate with respect to time t :

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}.$$

(d) Plug in and solve. At $h = 0.8$ m and $\frac{dV}{dt} = 1.5$ m³/min,

$$1.5 = \frac{\pi}{4} (0.8)^2 \frac{dh}{dt} = \frac{\pi}{4} (0.64) \frac{dh}{dt} = \frac{0.16\pi}{1} \frac{dh}{dt}.$$

Hence

$$\frac{dh}{dt} = \frac{1.5}{0.16\pi} = \frac{75}{8\pi} \text{ m/min} \approx 2.98 \text{ m/min}.$$

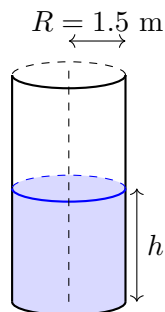
Since $r = \frac{1}{2}h$,

$$\frac{dr}{dt} = \frac{1}{2} \frac{dh}{dt} = \frac{75}{16\pi} \text{ m/min} \approx 1.49 \text{ m/min}.$$

$$\boxed{\frac{dh}{dt} = \frac{75}{8\pi} \text{ m/min } (\approx 2.98)}, \quad \boxed{\frac{dr}{dt} = \frac{75}{16\pi} \text{ m/min } (\approx 1.49)}.$$

3. A vertical cylindrical tank of constant radius 1.5 m has an open top; liquid drains so that the volume inside decreases at two tenths of a cubic meter per minute. Assuming the radius of the tank is fixed, how fast is the fluid level (height) falling when the depth is 2 m?

(a)



Given & Want.

$$r = 1.5 \text{ m}, \quad \frac{dV}{dt} = -0.2 \text{ m}^3/\text{min}.$$

Find $\frac{dh}{dt}$ when $h = 2 \text{ m}$.

(b) **Relation.** For a cylinder with fixed radius,

$$V = \pi r^2 h.$$

(c) **Differentiate.** Since r is constant,

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}.$$

(d) **Plug in & Solve.**

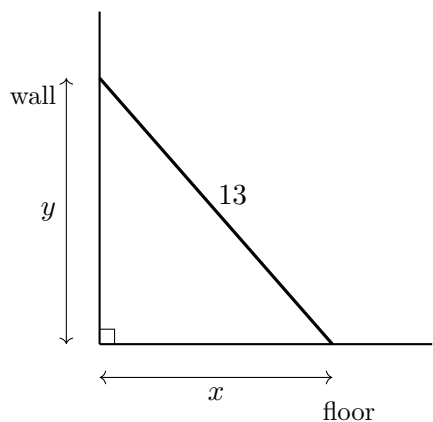
$$\frac{dh}{dt} = \frac{1}{\pi r^2} \frac{dV}{dt} = \frac{-0.2}{\pi(1.5)^2} = \frac{-0.2}{2.25\pi} = -\frac{4}{45\pi} \text{ m/min}$$

$$\boxed{\frac{dh}{dt} = -\frac{4}{45\pi} \text{ m/min} \approx -2.83 \text{ cm/min}}$$

4. Right Triangles: Ladders, Motion, and Shadows

1. A 13-ft ladder leans against a wall. The base slides away from the wall at 2 ft/s. How fast is the top sliding down when the base is 5 ft from the wall?

(a)



Given: $L = 13 \text{ ft}$; $x = 5 \text{ ft}$; $\frac{dx}{dt} = +2 \text{ ft/s}$.

Want: $\frac{dy}{dt}$ at that instant.

(b) Relationship (Pythagorean Theorem):

$$x^2 + y^2 = L^2 = 169.$$

(c) Differentiate with respect to time t :

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}.$$

(d) Compute y when $x = 5$: $y = \sqrt{169 - 25} = \sqrt{144} = 12$ ft. Then

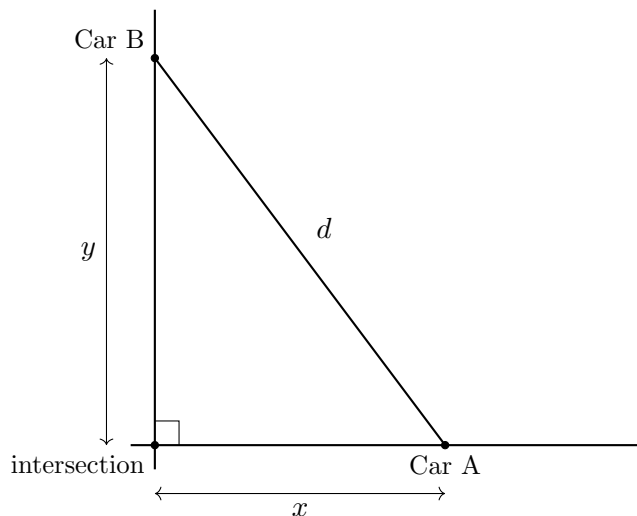
$$\frac{dy}{dt} = -\frac{5}{12} (2) = -\frac{5}{6} \text{ ft/s} \approx -0.833 \text{ ft/s}.$$

$\frac{dy}{dt} = -\frac{5}{6} \text{ ft/s}$

 The top is sliding down at 5/6 ft/s.

2. At a certain instant, Car A is 3 miles east of an intersection and moving east at 40 mph, while Car B is 4 miles north and moving north at 30 mph. At that instant, how fast is the distance between the cars changing?

(a)



Given: $x = 3$ mi, $\frac{dx}{dt} = 40$ mph; $y = 4$ mi, $\frac{dy}{dt} = 30$ mph.

Want: $\frac{dd}{dt}$ at that instant.

(b) Relationship (perpendicular motion):

$$x^2 + y^2 = d^2.$$

(c) Differentiate with respect to time t :

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2d \frac{dd}{dt} \implies \frac{dd}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{d}.$$

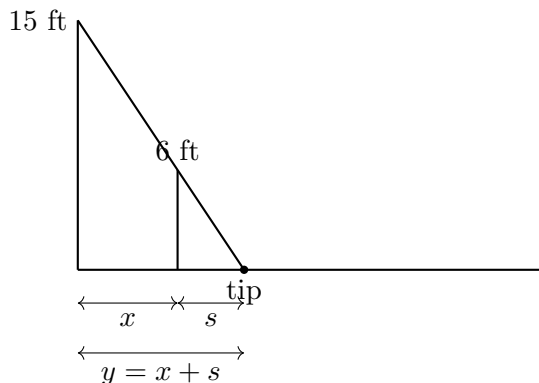
(d) Plug in and solve. First $d = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5$ mi.

$$\frac{dd}{dt} = \frac{(3)(40) + (4)(30)}{5} = \frac{120 + 120}{5} = 48 \text{ mph}.$$

$$\boxed{\frac{dd}{dt} = 48 \text{ mph}} \text{ — the distance between the cars is increasing at 48 mph.}$$

3. A 6-ft-tall person walks away from a 15-ft streetlamp at 3 ft/s. How fast is the tip of the shadow moving when the person is 20 ft from the lamp?

(a)



Variables: $x(t)$ =person's distance from lamp, $s(t)$ =shadow length, $y(t) = x + s$ =distance lamp to shadow tip.

Given: $x = 20$ ft, $\frac{dx}{dt} = 3$ ft/s, lamp = 15 ft, person = 6 ft.

Want: speed of tip = $\frac{dy}{dt}$.

(b) Relationship (similar triangles from lamp top to shadow tip):

$$\frac{15}{x + s} = \frac{6}{s} \implies 15s = 6(x + s) \Rightarrow 9s = 6x \Rightarrow s = \frac{2}{3}x.$$

$$\text{Hence } y = x + s = x + \frac{2}{3}x = \frac{5}{3}x.$$

(c) Differentiate with respect to time:

$$\frac{dy}{dt} = \frac{5}{3} \frac{dx}{dt}.$$

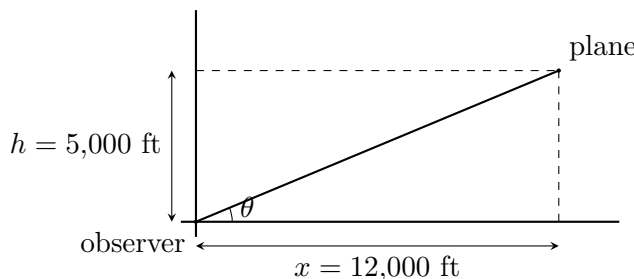
(d) Plug in and solve (at $x = 20$ ft, $\frac{dx}{dt} = 3$ ft/s):

$$\frac{dy}{dt} = \frac{5}{3} \cdot 3 = \boxed{5 \text{ ft/s}}.$$

5. Rotating Angles

1. An observer on level ground watches a plane flying in a straight line at a constant altitude of 5,000 ft. The plane is moving *away* from the observer with horizontal speed 400 ft/s. How fast is the angle of elevation changing when the plane is 12,000 ft horizontally from the observer?

(a)



Variables: $x(t)$ =horizontal distance, $h = 5,000$ ft, $\theta(t)$ =angle of elevation.

Given: $x = 12,000$ ft, $dx/dt = +400$ ft/s (plane receding).

Want: $d\theta/dt$.

(b) Relationship (right triangle):

$$\tan \theta = \frac{h}{x}.$$

(c) Differentiate with respect to time t :

$$\sec^2 \theta \frac{d\theta}{dt} = -\frac{h}{x^2} \frac{dx}{dt} \implies \frac{d\theta}{dt} = -\frac{h}{x^2 + h^2} \frac{dx}{dt},$$

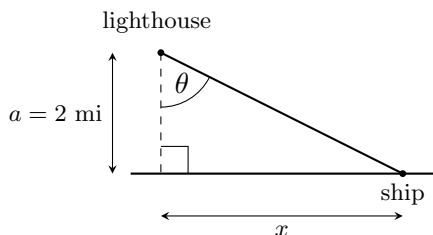
$$\text{using } \sec^2 \theta = 1 + \tan^2 \theta = \frac{x^2 + h^2}{x^2}.$$

(d) Plug in and solve:

$$\frac{d\theta}{dt} = -\frac{5,000}{(12,000)^2 + (5,000)^2} (400) = -\frac{2,000,000}{169,000,000} = \boxed{-\frac{2}{169} \text{ rad/s}}$$

2. A lighthouse stands 2 miles off a straight shoreline. A ship sails parallel to the shore at 12 mph, away from the lighthouse. How fast is the bearing angle from the lighthouse to the ship changing when the ship is 4 miles down the coast from the point closest to the lighthouse?

(a)



Variables: $x(t)$ = along-shore distance from the closest point; $\theta(t)$ = bearing angle at the lighthouse measured from the perpendicular to the shore.

Given at the instant: $a = 2$ mi (fixed), $x = 4$ mi, $\frac{dx}{dt} = 12$ mph.

Want: $\frac{d\theta}{dt}$.

(b) Relationship:

$$\tan \theta = \frac{x}{a} \quad \text{so} \quad x = a \tan \theta.$$

(c) Differentiate with respect to time:

$$\frac{dx}{dt} = a \sec^2 \theta \frac{d\theta}{dt} \implies \frac{d\theta}{dt} = \frac{1}{a \sec^2 \theta} \frac{dx}{dt} = \frac{a}{a^2 + x^2} \frac{dx}{dt},$$

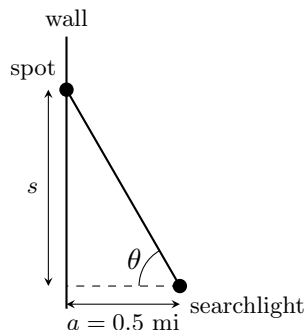
$$\text{since } \sec^2 \theta = 1 + \tan^2 \theta = 1 + \left(\frac{x}{a}\right)^2 = \frac{a^2 + x^2}{a^2}.$$

(d) Plug in and solve:

$$\frac{d\theta}{dt} = \frac{2}{2^2 + 4^2} (12) = \frac{2}{20} \cdot 12 = \frac{24}{20} = \boxed{\frac{6}{5} \text{ rad/hr}}$$

3. A searchlight located 0.5 miles from a straight wall rotates at a constant 0.30 radians per second. How fast is the light spot moving along the wall when the beam makes a 60° angle with the perpendicular to the wall?

(a)



Variables: $a = 0.5$ mi (fixed distance to wall), $\theta(t)$ = angle with the perpendicular, $s(t)$ = distance of the light spot along the wall from the foot of the perpendicular.

Given at the instant: $\theta = 60^\circ$, $\frac{d\theta}{dt} = 0.30$ rad/s.

Want: $\frac{ds}{dt}$ (speed of the light spot along the wall).

(b) Relationship (right triangle along the wall):

$$s = a \tan \theta.$$

(c) Differentiate with respect to time:

$$\frac{ds}{dt} = a \sec^2 \theta \frac{d\theta}{dt}.$$

(d) Plug in and solve (with $a = 0.5$ mi, $\theta = 60^\circ$, $\sec^2 60^\circ = 4$, $\frac{d\theta}{dt} = 0.30$ rad/s):

$$\frac{ds}{dt} = (0.5)(4)(0.30) = 0.60 \text{ mi/s}.$$

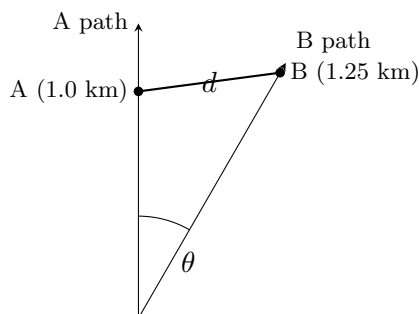
Converting units: $0.60 \text{ mi/s} \times 3600 = 2160 \text{ mph}$.

$\frac{ds}{dt} = 0.60 \text{ miles per second } (\approx 2160 \text{ mph})$

6. Mixed Motion (Non-Perpendicular Paths)

- Two hikers leave the same point; Hiker A walks straight north at 4 km/h. Hiker B walks at 5 km/h on a path turning so that the angle between their paths increases at $2^\circ/\text{min}$. How fast is their distance changing after 15 minutes?

(a)



Given at $t = 15$ min:

Distances along paths:

$$a = (4 \text{ km/h}) \cdot (0.25 \text{ h}) = 1.0 \text{ km},$$

$$b = (5 \text{ km/h}) \cdot (0.25 \text{ h}) = 1.25 \text{ km}.$$

Angle between paths:

$$\theta = (2^\circ/\text{min}) \cdot 15 \text{ min} = 30^\circ,$$

$$\frac{d\theta}{dt} = \frac{2\pi}{180} \text{ rad/min} = \frac{\pi}{90} \text{ rad/min}.$$

Speeds (in km/min):

$$\frac{da}{dt} = \frac{4}{60} = \frac{1}{15}$$

$$\frac{db}{dt} = \frac{5}{60} = \frac{1}{12}$$

Want: $\frac{dd}{dt}$ when $t = 15$ min.

(b) Relationship (Law of Cosines):

$$d^2 = a^2 + b^2 - 2ab \cos \theta.$$

(c) Differentiate with respect to time t (all functions of t):

$$2d \frac{dd}{dt} = 2a a' + 2b b' - 2(a'b + ab') \cos \theta + 2ab \sin \theta \frac{d\theta}{dt}.$$

Hence

$$\frac{dd}{dt} = \frac{a a' + b b' - (a'b + ab') \cos \theta + ab \sin \theta \frac{d\theta}{dt}}{d}.$$

(d) Plug in $a = 1$, $b = \frac{5}{4}$, $a' = \frac{1}{15}$, $b' = \frac{1}{12}$, $\theta = 30^\circ$ (so $\cos \theta = \frac{\sqrt{3}}{2}$, $\sin \theta = \frac{1}{2}$), and $\frac{d\theta}{dt} = \frac{\pi}{90}$. First compute d :

$$d = \sqrt{1^2 + \left(\frac{5}{4}\right)^2 - 2 \cdot 1 \cdot \frac{5}{4} \cdot \frac{\sqrt{3}}{2}} = \sqrt{\frac{41}{16} - \frac{5\sqrt{3}}{4}} \approx 0.6304 \text{ km}.$$

Then the numerator:

$$aa' + bb' - (a'b + ab') \cos \theta + ab \sin \theta \frac{d\theta}{dt} \approx 0.04831 \text{ km/min}.$$

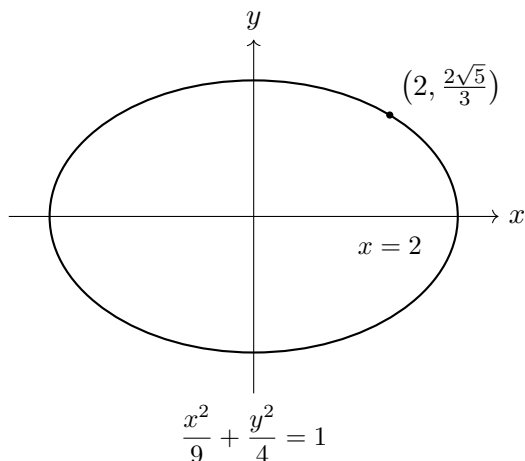
Therefore

$$\frac{dd}{dt} \approx \frac{0.04831}{0.6304} = 0.0766 \text{ km/min} = \boxed{\approx 4.60 \text{ km/h}}$$

7. Motion Constrained to a Curve

1. A point moves on $\frac{x^2}{9} + \frac{y^2}{4} = 1$. When it is at $(2, \frac{2\sqrt{5}}{3})$, the x -coordinate increases at 0.3 units/s. How fast is y changing?

(a)



Given: $\frac{x^2}{9} + \frac{y^2}{4} = 1$, $x = 2$, $y = \frac{2\sqrt{5}}{3}$, $\frac{dx}{dt} = 0.3$ units/s.

Want: $\frac{dy}{dt}$ at that instant.

(b) Relationship:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

(c) Differentiate with respect to t :

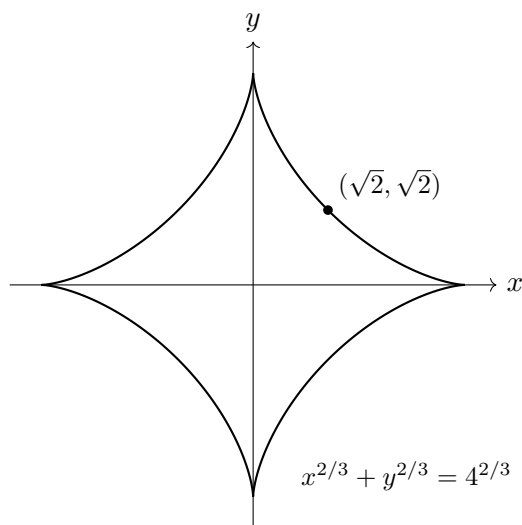
$$\frac{2x}{9} \frac{dx}{dt} + \frac{2y}{4} \frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{4x}{9y} \frac{dx}{dt}.$$

(d) Plug in $x = 2$, $y = \frac{2\sqrt{5}}{3}$, $\frac{dx}{dt} = 0.3$:

$$\frac{dy}{dt} = -\frac{4(2)}{9\left(\frac{2\sqrt{5}}{3}\right)}(0.3) = -\frac{8}{6\sqrt{5}}(0.3) = -\frac{4}{3\sqrt{5}}(0.3) = -\frac{0.4}{\sqrt{5}} = -\frac{2}{5\sqrt{5}} = \boxed{-\frac{2\sqrt{5}}{25} \text{ units/s.}}$$

2. A bead slides on the curve $x^{2/3} + y^{2/3} = 4^{2/3}$. At the point $(\sqrt{2}, \sqrt{2})$, the horizontal speed is 1 cm/s to the right. Find the vertical speed (state up or down).

(a)



Given: $x^{2/3} + y^{2/3} = 4^{2/3}$, $x = \sqrt{2}$, $y = \sqrt{2}$, $\frac{dx}{dt} = +1$ cm/s.

Want: $\frac{dy}{dt}$ at $(\sqrt{2}, \sqrt{2})$.

(b) Relationship (implicit equation of the path):

$$x^{2/3} + y^{2/3} = 4^{2/3}.$$

(c) Differentiate both sides with respect to t :

$$\frac{2}{3}x^{-1/3}\frac{dx}{dt} + \frac{2}{3}y^{-1/3}\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{x^{-1/3}}{y^{-1/3}}\frac{dx}{dt} = -\left(\frac{y}{x}\right)^{1/3}\frac{dx}{dt}.$$

(d) Plug in and solve. At $(x, y) = (\sqrt{2}, \sqrt{2})$, we have $\left(\frac{y}{x}\right)^{1/3} = 1$, so

$$\frac{dy}{dt} = -1 \cdot (1 \text{ cm/s}) = -1 \text{ cm/s}.$$

$\frac{dy}{dt} = -1 \text{ cm/s}$

 — the bead is moving *down*.