Midterm 1 Study Guide – Solutions

MATH1300 - Calculus I

Fall 2025

Contents

1.1–1.3 Precalculus Review

Lines

1. A line parallel to $y = -3x + 7$ has slope -3. Through $(2, 4)$, the equation is $y = -3x + 10$.

2. A line perpendicular to $y = \frac{1}{2}x - 5$ has slope -2. Through $(0, 6)$, the equation is $y = -2x + 6$.

3. The slope through $(-1, 2)$ and $(3, -4)$ is $-3/2$. The line is $y = -\frac{3}{2}x + \frac{1}{2}$.

4. A line with slope -2 through $(5, 1)$ can be written in point-slope form $y - 1 = -2(x - 5)$ or slope-intercept form $y = -2x + 11$.

Polynomial End Behavior

5. For $f(x) = -4x^5 + 2x^2 - 7$, the leading term is odd degree with a negative coefficient. As $x \to \infty$, $f(x) \to -\infty$, and as $x \to -\infty$, $f(x) \to \infty$.

6. For $f(x) = 3x^4 - 5x^3 + x - 1$, the leading term is even degree with a positive coefficient. As $x \to \pm\infty$, $f(x) \to \infty$.

7. For $f(x) = -x^{10} + 8x^7 - 2$, the leading term is even degree with a negative coefficient. As $x \to \pm\infty$, $f(x) \to -\infty$.

Logs & Exponentials

8. $\log_5(125) = 3$ since $5^3 = 125$.

9. $\log_2(16) - \log_2(4) = 4 - 2 = 2$.

10. To solve $3^x = 81$, note that $81 = 3^4$. Thus $x = 4$.

Difference Quotient Practice

11. For $f(x) = x^2 + 3x$,

$$\begin{aligned}\frac{f(a+h) - f(a)}{h} &= \frac{(a+h)^2 + 3(a+h) - (a^2 + 3a)}{h} \\ &= \frac{2ah + h^2 + 3h}{h} \\ &= 2a + h + 3.\end{aligned}$$

12. For $f(x) = \sqrt{x+1}$,

$$\begin{aligned}\frac{f(2+h) - f(2)}{h} &= \frac{\sqrt{3+h} - \sqrt{3}}{h} \cdot \frac{\sqrt{3+h} + \sqrt{3}}{\sqrt{3+h} + \sqrt{3}} \\ &= \frac{1}{\sqrt{3+h} + \sqrt{3}}.\end{aligned}$$

Polynomial Algebra

13. The polynomial $P(x) = -6x^2 + 4x - 7$ is in standard form. It has degree 2, coefficients $-6, 4, -7$, leading coefficient -6, and terms $-6x^2, 4x, -7$.

14. For $P(x) = 3x^4 - 2x^3 + x - 5$, the degree is 4, the leading coefficient is 3, and the constant term is -5.

15. Expanding $Q(x) = (x-1)(x+2)(x-3)$ gives $x^3 - 2x^2 - 5x + 6$. The degree is 3 and the leading coefficient is 1.

Polynomial Equations & Factoring

16. The equation $x^2 - 5x + 6 = 0$ factors as $(x-2)(x-3) = 0$. The solutions are $x = 2$ and $x = 3$.

17. Solving $2x^2 + 7x + 3 = 0$ gives

$$x = \frac{-7 \pm \sqrt{49 - 24}}{4} = \frac{-7 \pm 5}{4},$$

so $x = -\frac{1}{2}$ and $x = -3$.

18. The difference of cubes $x^3 - 27$ factors as $(x-3)(x^2 + 3x + 9)$.

Trig Review

19. $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, and $\tan\left(\frac{\pi}{4}\right) = 1$.

20. $\csc\left(\frac{\pi}{2}\right) = 1$, $\sec(0) = 1$, and $\cot\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{3}$.

21. For $y = 3\sin(2x)$, the amplitude is 3 and the period is π.

22. For $y = -\frac{1}{2}\cos(\pi x)$, the amplitude is $\frac{1}{2}$ and the period is 2.

2.1 Tangent & Velocity Problems

Average Velocity (Data Tables)

1. On $[1, 3]$:

$$v_{avg} = \frac{s(3) - s(1)}{3 - 1} = \frac{15 - 2}{2} = 6.5 \text{ m/s}.$$

On $[2, 4]$:

$$v_{avg} = \frac{26 - 7}{2} = 9.5 \text{ m/s}.$$

2. On $[0, 2]$:

$$v_{avg} = \frac{s(2) - s(0)}{2 - 0} = \frac{11 - 20}{2} = -4.5 \text{ m/s}.$$

On $[2, 4]$:

$$v_{avg} = \frac{6 - 11}{2} = -2.5 \text{ m/s}.$$

Average Velocity (Formulas)

3. $s(t) = t^2 + 3t$ (m).

$$[2, 5]: \ v_{avg} = \frac{s(5) - s(2)}{3} = \frac{40 - 10}{3} = 10 \text{ m/s},$$
$$[5, 6]: \ v_{avg} = \frac{s(6) - s(5)}{1} = 54 - 40 = 14 \text{ m/s}.$$

4. $s(t) = \sqrt{t + 4}$ (m).

$$[0, 1]: \ v_{avg} = \sqrt{5} - 2 \ \approx 0.236 \text{ m/s},$$
$$[1, 4]: \ v_{avg} = \frac{\sqrt{8} - \sqrt{5}}{3} \ \approx 0.197 \text{ m/s}.$$

Secant Slopes and Tangent Slope

5. For $f(x) = x^2$, the secant slope between $(2, f(2))$ and $(2 + h, f(2 + h))$ is

$$m = \frac{(2 + h)^2 - 4}{h} = \frac{4 + 4h + h^2 - 4}{h} = 4{+}h.$$

h	1	0.5	0.1	0.01	-0.01	-0.1
slope $(4 + h)$	5	4.5	4.1	4.01	3.99	3.9

Estimate: the tangent slope at $x = 2$ is $\boxed{4}$.

Average Rates from Real-World Data

6. Car trip:

$$[0, 3]: v_{avg} = \frac{150}{3} = 50 \text{ mph},$$
$$[3, 4]: v_{avg} = \frac{60}{1} = 60 \text{ mph},$$
$$[0, 4]: v_{avg} = \frac{210}{4} = 52.5 \text{ mph}.$$

7. $s(t) = 100t - 5t^2$ (m).

$$\begin{aligned} [2, 3]: \ v_{avg} &= \frac{(300 - 45) - (200 - 20)}{1} \\ &= 75 \text{ m/s}, \\ [2, 2.1]: \ v_{avg} &= \frac{(210 - 22.05) - (200 - 20)}{0.1} \\ &= 79.5 \text{ m/s}. \end{aligned}$$

This suggests the instantaneous velocity at $t = 2$ is about $\boxed{80 \text{ m/s}}$.

8. Turkey cooling (°F per min). Secant on $[55, 60]$:

$$\frac{109 - 114}{5} = -1.0.$$

Secant on $[60, 65]$:

$$\frac{105 - 109}{5} = -0.8.$$

Average these to estimate the instantaneous rate at $t = 60$:

$$\boxed{-0.9 \text{ °F/min}}.$$

9. Reservoir depth $W(x)$ (m). On $[150, 250]$:

$$\bar{r} = \frac{W(250) - W(150)}{250 - 150} = \frac{54 - 50}{100} = 0.04 \text{ m/day}.$$

Interpretation: between day 150 and 250, the water depth increased on average by $\boxed{0.04 \text{ m per day}}$.

2.2 Limits

One-Sided Limits

1. $\lim_{x\to 0^-} \frac{1}{x} = -\infty$, $\lim_{x\to 0^+} \frac{1}{x} = +\infty$.
2. $\lim_{x\to(\pi/2)^-} \tan x = +\infty$, $\lim_{x\to(\pi/2)^+} \tan x = -\infty$.

Absolute Value Limits

3. For $x < 0$, $|x| = -x$ so $\frac{|x|}{x} = -1$; for $x > 0$, $\frac{|x|}{x} = 1$:
   - $\lim_{x\to 0^-} \frac{|x|}{x} = -1$,
   - $\lim_{x\to 0^+} \frac{|x|}{x} = 1$
4. For $x < 2$, $|x-2| = -(x-2)$ so $\frac{|x-2|}{x-2} = -1$; for $x > 2$, it equals 1:
   - $\lim_{x\to 2^-} \frac{|x-2|}{x-2} = -1$
   - $\lim_{x\to 2^+} \frac{|x-2|}{x-2} = 1$

Piecewise Functions

5. $f(x) = \begin{cases} 2x+1 & x < 1 \\ 5 & x = 1 \\ x^2 & x > 1 \end{cases}$
   - $\lim_{x\to 1^-} f(x) = 2(1) + 1 = 3$,
   - $\lim_{x\to 1^+} f(x) = (1)^2 = 1$,
   - $\lim_{x\to 1} f(x) = \text{DNE}$.
6. $g(x) = \begin{cases} x^2 & x \le 0 \\ \sqrt{x} & x > 0 \end{cases}$
   - $\lim_{x\to 0^-} g(x) = 0^2 = 0$,
   - $\lim_{x\to 0^+} g(x) = \sqrt{0} = 0$,
   - $\lim_{x\to 0} g(x) = 0$.

Graph-Based Interpretation

7. From the graph:
   - $\lim_{x\to -2^-} f(x) = 1$,
   - $\lim_{x\to -2^+} f(x) = 3$,
   - $\lim_{x\to -2} f(x)$ DNE,
   - $f(-2) = 3$.
8. From the graph:
   - $\lim_{x\to 2^-} f(x) = -\infty$,
   - $\lim_{x\to 2^+} f(x) = +\infty$,
   - $\lim_{x\to 2} f(x)$ does not exist.
9. From the graph:
   - $\lim_{x\to 1^-} f(x) = 2$,
   - $\lim_{x\to 1^+} f(x) = 2$,
   - $\lim_{x\to 1} f(x) = 2$,
   - $f(1) = -1$.
10. From the graph:
    - $\lim_{x\to 0^-} f(x) = -1$,
    - $\lim_{x\to 0^+} f(x) = 2$,
    - $\lim_{x\to 0} f(x)$ DNE,
    - $f(0) = 1$.

2.3 Limit Laws

Direct Use of Limit Laws

1.

$$\begin{aligned}
&\lim_{x\to 2}\left(3x^2-4x+7\right)\\
&=\lim_{x\to 2}(3x^2)\ -\ \lim_{x\to 2}(4x)\ +\ \lim_{x\to 2}(7)\\
&=3\lim_{x\to 2}(x^2)\ -\ 4\lim_{x\to 2}(x)\ +\ 7\lim_{x\to 2}(1)\\
&=3\cdot(2^2)\ -\ 4\cdot(2)\ +\ 7\cdot(1)\\
&=12-8+7\\
&=\boxed{11}.
\end{aligned}$$

2.

$$\begin{aligned}
&\lim_{x\to -1}\left(x^3+2x^2-5x\right)\\
&=\lim_{x\to -1}(x^3)\ +\ \lim_{x\to -1}(2x^2)\ -\ \lim_{x\to -1}(5x)\\
&=\lim_{x\to -1}(x^3)\ +\ 2\lim_{x\to -1}(x^2)\ -\ 5\lim_{x\to -1}(x)\\
&=(-1)^3+2(-1)^2-5(-1)\\
&=-1+2+5\\
&=\boxed{6}.
\end{aligned}$$

3.

$$\begin{aligned}
\lim_{x\to 4}\left(x^2+\sqrt{x}\right)&=\lim_{x\to 4}(x^2)\ +\ \lim_{x\to 4}(\sqrt{x})\\
&=\left(\lim_{x\to 4}x\right)^2+\sqrt{\lim_{x\to 4}x}\\
&=4^2+\sqrt{4}\\
&=16+2\\
&=\boxed{18}.
\end{aligned}$$

Rational Functions

4.

$$\begin{aligned}
\lim_{x\to 3}\frac{x^2-9}{x-3}&=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}\\
&=\lim_{x\to 3}(x+3)\\
&=3+3=\boxed{6}.
\end{aligned}$$

5.

$$\begin{aligned}
\lim_{x\to -2}\frac{x^3+8}{x+2}&=\lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{x+2}\\
&=\lim_{x\to -2}(x^2-2x+4)\\
&=4+4+4\\
&=\boxed{12}.
\end{aligned}$$

Roots

6.

$$\begin{aligned}
\lim_{x\to 15}\sqrt[4]{x+1}&=\sqrt[4]{\lim_{x\to 15}(x+1)}\\
&=\sqrt[4]{16}\\
&=\boxed{2}.
\end{aligned}$$

7.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}&=\lim_{x\to 0}\frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}\\
&=\lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2)}\\
&=\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}\\
&=\boxed{\tfrac{1}{4}}.
\end{aligned}$$

8.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}&=\lim_{x\to 0}\frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})}\\
&=\lim_{x\to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\\
&=\boxed{1}.
\end{aligned}$$

9.

$$\begin{aligned}
\lim_{x\to 1}\frac{\sqrt{5x+4}-3}{x-1}&=\lim_{x\to 1}\frac{(5x+4)-9}{(x-1)(\sqrt{5x+4}+3)}\\
&=\lim_{x\to 1}\frac{5(x-1)}{(x-1)(\sqrt{5x+4}+3)}\\
&=\lim_{x\to 1}\frac{5}{\sqrt{5x+4}+3}\\
&=\boxed{\tfrac{5}{6}}.
\end{aligned}$$

Squeeze Theorem

10. We have:

$$x - 1 \;\le\; h(x) \;\le\; x^2 + x - 2.$$

As $x \to 1$, both bounds go to 0. Hence,

$$\lim_{x\to 1} h(x) = \boxed{0}.$$

by the Squeeze Theorem.

11. Because $-1 \le \sin(1/x) \le 1$, multiplying through by $x^2 \ge 0$ gives

$$-x^2 \;\le\; x^2 \sin\left(\tfrac{1}{x}\right) \;\le\; x^2.$$

As $x \to 0$, both bounds go to 0, so by the Squeeze Theorem

$$\lim_{x\to 0} x^2 \sin\left(\tfrac{1}{x}\right) = \boxed{0}.$$

12. For $x \neq 0$,

$$-x^2 \;\le\; x^2 e^{-1/x^2} \sin\left(\tfrac{1}{x}\right) \;\le\; x^2.$$

As $x \to 0$, both bounds tend to 0, so

$$\lim_{x\to 0} x^2 e^{-1/x^2} \sin\left(\tfrac{1}{x}\right) = \boxed{0}.$$

Limits from Graphs

13. Using the plotted behaviors of f and g:

(a) As $x \to 5^-$: $f(x) \to 4$ and $g(x) \to -2$. Thus

$$\lim_{x\to 5^-} (f+g) = 4 + (-2) = \boxed{2}.$$

(b) Near $x = 3$: $f(x) \to 2$ from the left and $f(x) \to 0$ from the right. Meanwhile $g(x) \to 0$ from the left and $g(x) \to -2$ from the right. Hence

- $\lim_{x\to 3^-}(f-g) = 2 - 0 = 2$,
- $\lim_{x\to 3^+}(f-g) = 0 - (-2) = 2$
- $\boxed{\lim_{x\to 3}(f-g) = 2}$.

(c) Near $x = 1$: $f(x) \to -2$ from the left and $f(x) \to 2$ from the right. Then $f^2 \to (-2)^2 = 4$ from the left and $f^2 \to 2^2 = 4$ from the right, so

$$\boxed{\lim_{x\to 1} f^2 = 4}.$$

(d) Since $\lim_{x\to -2} g = -4$,

$$\boxed{\lim_{x\to -2} (5g) = -20}.$$

(e) As $x \to 0$: $f(x) \to 0$ while $g(x)$ tends to a nonzero value. The quotient blows up with opposite signs from the two sides, so

$$\boxed{\lim_{x\to 0} \left(\frac{g}{f}\right) = \text{DNE}}.$$

(f) As $x \to -1$: $g(x) \to 0$, while $f(x) \to 4$ from the left and $f(x) \to 2$ from the right. The ratio tends to $-\infty$ from both sides:

$$\boxed{\lim_{x\to -1} \left(\frac{f}{g}\right) = -\infty}.$$

(g) As $x \to -4$: $f(x) \to -4$ from the left and $f(x) \to 3$ from the right; $g(x) \to 3$ from the left and $g(x) \to -4$ from the right . Both one-sided products approach $(-4)\cdot 3 = 3\cdot(-4) = -12$, so

$$\boxed{\lim_{x\to -4} (fg) = -12}.$$

2.5 Continuity & IVT

Continuity at a Point

1. Note:

$$f(x) = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$

for $x \neq 2$. In particular, $\lim_{x\to 2} f(x) = 4$, but $f(2)$ is undefined. Therefore, $f(x)$ is not continuous at $x = 2$.

2. $\tan x$ has a vertical asymptote at $x = \frac{\pi}{2}$, and the one-sided limits are $\pm\infty$. It is not continuous at $x = \frac{\pi}{2}$.

Piecewise Continuity

3. For $f(x) = \begin{cases} x^2+1, & x<2 \\ ax+b, & x \geq 2 \end{cases}$, continuity at $x = 2$ requires

$$\lim_{x\to 2^-} f(x) = 2^2+1 = 5 = (2a+b) = f(2).$$

Solutions: any a, b with $2a+b = 5$. For example, $a = 2,\ b = 1$.

4. For $g(x) = \begin{cases} \sqrt{x+1}, & x>0 \\ k, & x=0 \\ x^2+1, & x<0 \end{cases}$,

$$\lim_{x\to 0^-} g = 0^2+1 = 1, \qquad \lim_{x\to 0^+} g = \sqrt{1} = 1.$$

Take $k = 1$ for continuity at 0.

5. $F(x) = \dfrac{x^2-9}{x-3}$ for $x \neq 3$ equals $x+3$. Thus $\lim_{x\to 3} F(x) = 6$ so choose $c = \boxed{6}$.

6. For $G(x) = \dfrac{\sqrt{x-1}-2}{x-5}$ at $x = 5$, rationalize:

$$\frac{\sqrt{x-1}-2}{x-5} \cdot \frac{\sqrt{x-1}+2}{\sqrt{x-1}+2} = \frac{x-5}{(x-5)\left(\sqrt{x-1}+2\right)} = \frac{1}{\sqrt{x-1}+2}.$$

Hence $\lim_{x\to 5} G = \dfrac{1}{\sqrt{4}+2} = \boxed{\frac{1}{4}}$. Set $c = \frac{1}{4}$.

Classifying Discontinuities

7. $\dfrac{x^2-1}{x-1} = \dfrac{(x-1)(x+1)}{x-1} = x+1$ for $x \neq 1$. Therefore, there is a hole at $x = 1$. $\boxed{\text{removable}}$.

8. $\dfrac{1}{x^2}$ blows up at $x = 0$. $\boxed{\text{infinite}}$.

9. $\dfrac{|x|}{x}$ jumps from -1 to 1 at 0. $\boxed{\text{jump}}$.

Graph-Based Reasoning

10. From the graph:

$$\begin{aligned} x=-2 &: \boxed{\text{jump}} \\ x=0 &: \boxed{\text{jump}} \\ x=1 &: \boxed{\text{infinite}} \\ x=2 &: \boxed{\text{jump}} \\ x=3 &: \boxed{\text{removable}} \end{aligned}$$

Intermediate Value Theorem

11. $f(x) = x^3 - 5x + 2$ is continuous, since it is a polynomial. Check values:

$$f(0) = 2 > 0, \qquad f(1) = 1-5+2 = -2 < 0.$$

By the IVT, there exists $0 < c < 1$ with $f(c) = 0$.

12. Consider $F(x) = \cos x - x$ on $[0, 1]$. This is continuous, as it is a difference of two continuous functions, and

$$F(0) = 1 > 0, \qquad F(1) = \cos 1 - 1 < 0.$$

By IVT, there exists $0 < c < 1$ with $\cos c = c$.

13. $f(x) = e^x - 4$ is continuous; $f(1) = e - 4 < 0$, $f(2) = e^2 - 4 > 0$. By IVT, a root lies in $(1, 2)$.

Intervals of Continuity

14. For $h(x) = \begin{cases} \dfrac{1}{x+2}, & x < -2 \\ x^2 - 1, & -2 \le x < 1 \\ \sqrt{x-1}, & x \ge 1 \end{cases}$:

$$\text{Continuous on } (-\infty, -2),\ [-2, \infty).$$

At $x = -2$: jump discontinuity. At $x = 1$ it is continuous.

15. $p(x) = \ln(x-3) + \dfrac{x}{x-5}$.

$$\text{Domain: } (3,5) \cup (5, \infty).$$

Continuous on $(3,5)$ and $(5,\infty)$. At $x = 5$, there is a vertical asymptote.

Continuity Theorems & Compositions

16. We have

$$\begin{aligned} g(4) &= g(f(0)) && \text{since } f(0) = 4 \\ &= g\big(\lim_{x\to 0} f(x)\big) && \text{since } f \text{ is continuous at } x = 0 \\ &= \lim_{x\to 0} g\big(f(x)\big) && \text{since } g \text{ is continuous at } x = 4 \\ &= 7 \end{aligned}$$

17. The function $h(x)$ is continuous at $x = a$ because it is the composition of two continuous functions. That is, $f(x)$ is continuous at $x = a$ and $\sqrt{x}$ is continuous at $x = 9$.

$$\lim_{x\to a} h(x) = \lim_{x\to a} \sqrt{f(x)} = \sqrt{\lim_{x\to a} f(x)} = \sqrt{f(a)} = \boxed{3}.$$

Repairing Removable Discontinuities

18. $f(x) = \dfrac{x^2-1}{x-1} = x + 1$ for $x \ne 1$. Define

$$g(x) = \begin{cases} x+1, & x \ne 1 \\ 2, & x = 1 \end{cases}$$

so g is continuous and $g(1) = \boxed{2}$.

19. $\dfrac{x^3-8}{x-2} = \dfrac{(x-2)(x^2+2x+4)}{(x-2)} = x^2+2x+4$ for $x \ne 2$. Define

$$g(x) = \begin{cases} x^2+2x+4, & x \ne 2 \\ 12, & x = 2 \end{cases}$$

so g is continuous and $g(2) = \boxed{12}$.

2.6 Limits at Infinity & Asymptotes

Horizontal Asymptotes (Rational)

1. The degrees of the numerator and the denominator are the same, so use leading–coefficient ratio: horizontal asymptote $y = \frac{3}{2}$.
2. The numerator's degree is one larger, so there is no horizontal asymptote; the function grows like $5x$ to $\pm\infty$.
3. Since the denominator has a higher power than the numerator, both limits are 0: $\lim_{x\to\pm\infty} \frac{2x+1}{x^3+4} = 0$.
4. The degrees of the numerator and the denominator are the same, so the end behavior tends to the leading–coefficient ratio: $y = -2$ (horizontal asymptote on both ends).

Vertical Asymptotes (Rational)

5. Denominator $x^2-9 = (x-3)(x+3)$ vanishes at $\boxed{x = \pm 3}$; numerator $(x-1)$ is nonzero there $\Rightarrow$ vertical asymptotes at $x = -3, 3$.
6. Denominator is $(x-2)^2 \Rightarrow$ vertical asymptote at $\boxed{x = 2}$.
7. Denominator $x(x-1)$ zeros are $\boxed{x = 0, 1}$ (numerator is nonzero) $\Rightarrow$ vertical asymptotes at $x = 0, 1$.

Vertical Asymptotes (Log/Trig)

8. $\ln x$ is defined for $x > 0$ and $\ln x \to -\infty$ as $x \to 0^+ \Rightarrow$ vertical asymptote at $\boxed{x = 0}$.
9. $\tan x = \frac{\sin x}{\cos x}$ blows up where $\cos x = 0$. In $(-\pi, \pi)$: $\boxed{x = -\frac{\pi}{2},\ \frac{\pi}{2}}$.

Oblique (Slant) Asymptotes

10. Divide: $\frac{2x^2-3x+1}{x-1} = 2x - 1$ exactly, with a hole at $x = 1$ (since $(x-1)$ cancels). Slant line $y = 2x - 1$. No vertical asymptote.
11. Long division gives $f(x) = x+1+\frac{x}{x^2-1}$. As $x \to \pm\infty$, the fraction goes to 0. Hence there is a slant asymptote at $\boxed{y = x + 1}$. The denominator $(x-1)(x+1)$ vanishes at $x = \pm 1$ and the numerator is nonzero there, so there are vertical asymptotes at $\boxed{x = -1, 1}$.

Horizontal Asymptotes (Non-Rational)

12. $\arctan x \to \pm\frac{\pi}{2}$ as $x \to \pm\infty$. Horizontal asymptotes: $\boxed{y = \pm\frac{\pi}{2}}$.
13. As $x \to \infty$, $e^{-x} \to 0$, which means $f \to 1$. As $x \to -\infty$, $e^{-x} \to \infty$, and so $f \to 0$. Horizontal asymptotes: $\boxed{y = 0 \text{ and } y = 1}$.
14. Start by rationalizing:

$$\sqrt{x^2+1} - x = \frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}$$
$$= \frac{1}{\sqrt{x^2+1}+x}.$$

For $x > 0$,

$$\lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{1}{x\left(\sqrt{1+\frac{1}{x^2}}+1\right)}$$
$$= 0.$$

For $x < 0$, write $\sqrt{x^2+1} = (-x)\sqrt{1+\frac{1}{x^2}}$. Then

$$\lim_{x\to-\infty} \frac{1}{\sqrt{x^2+1}+x} = \lim_{x\to-\infty} \frac{1}{(-x)\left(\sqrt{1+\frac{1}{x^2}}-1\right)}$$
$$= \lim_{x\to-\infty} \frac{\sqrt{1+\frac{1}{x^2}}+1}{(-x)\left(\frac{1}{x^2}\right)}$$
$$= \lim_{x\to-\infty} -x\left(\sqrt{1+\frac{1}{x^2}}+1\right)$$
$$\infty.$$

Interpretation. As $x \to \infty$, the graph approaches the horizontal line $y = 0$ (right-end horizontal asymptote). As $x \to -\infty$, the function grows without bound, so there is no left-end horizontal asymptote.

15.

$$\begin{aligned} x - \sqrt{x^2+2x} &= \frac{(x-\sqrt{x^2+2x})(x+\sqrt{x^2+2x})}{x+\sqrt{x^2+2x}} \\ &= \frac{x^2-(x^2+2x)}{x+\sqrt{x^2+2x}} \\ &= \frac{-2x}{x+\sqrt{x^2+2x}} \\ &= \frac{-2}{1+\sqrt{1+\frac{2}{x}}} \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{x\to\infty}\left(x-\sqrt{x^2+2x}\right) &= \lim_{x\to\infty}\frac{-2}{1+\sqrt{1+\frac{2}{x}}} \\ &= \frac{-2}{1+1} \\ &= -1 \end{aligned}$$

Interpretation. The right-end horizontal asymptote is $y = -1$. Hence $y = 0$ is *not* a horizontal asymptote.

Growth-Rate Comparisons (Poly vs. Exp vs. Log)

16. $\lim_{x\to\infty}\frac{\ln x}{x} = 0$ (polynomial in the denominator wins); $\lim_{x\to\infty}\frac{x}{\ln x} = \infty$.
17. Exponential beats any fixed power: $\lim_{x\to\infty}\frac{e^x}{x^5} = \infty$, $\lim_{x\to\infty}\frac{x^7}{e^{0.1x}} = 0$.
18. Let $t = \sqrt{x}$, so that $x = t^2$. Then $\frac{x^3 \ln x}{e^{\sqrt{x}}} = \frac{t^6\,(2\ln t^2)}{e^t}$, which goes to 0. The exponential $e^{\sqrt{x}}$ dominates.
19. $\lim_{x\to\infty}\frac{(\ln x)^4}{x^{1/3}} = 0$. The (even small) power $x^{1/3}$ grows faster than any power of $\ln x$.

Vertical Asymptotes (One-Sided Sign Analysis)

20. $\lim_{x\to -6^-}\frac{x+5}{x+6} = +\infty$ (negative over tiny negative), $\lim_{x\to -6^+}\frac{x+5}{x+6} = -\infty$ (negative over tiny positive). Two-sided limit does not exist; vertical asymptote at $\boxed{x=-6}$ with opposite signs on the two sides.
21. $\lim_{x\to 2^-}\frac{1}{x-2} = -\infty$, $\lim_{x\to 2^+}\frac{1}{x-2} = +\infty$. In contrast, $\lim_{x\to 2}\frac{1}{(x-2)^2} = +\infty$ from *both* sides.
22. For $f(x) = \frac{2x-1}{(x+3)^2}$, the denominator is always positive and $\to 0$ as $x \to -3$, while the numerator $\to -7$. Thus $\lim_{x\to -3^\pm} f(x) = -\infty$. Vertical asymptote at $\boxed{x=-3}$ with negative blow-up on both sides.

Holes vs. Vertical Asymptotes

23. $\frac{x^2+2x-3}{x^2-9} = \frac{(x+3)(x-1)}{(x+3)(x-3)} = \frac{x-1}{x-3}$ for $x \neq -3$. Hole at $\boxed{x=-3}$; vertical asymptote at $\boxed{x=3}$.
24. $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$ for $x \neq 1$. No vertical asymptotes; there is a hole at $\boxed{x=1}$ (the point $(1,2)$ is removed).
25. $\frac{x^2-4}{x^2-x-2} = \frac{(x-2)(x+2)}{(x-2)(x+1)} = \frac{x+2}{x+1}$ for $x \neq 2$. Hole at $\boxed{x=2}$; vertical asymptote at $\boxed{x=-1}$.

2.7 Derivatives & Rates of Change

Definition of the Derivative

1.

$$f'(2) = \lim_{h\to 0} \frac{(2+h)^2 - 4}{h}$$
$$= \lim_{h\to 0} \frac{4 + 4h + h^2 - 4}{h}$$
$$= \lim_{h\to 0} (4+h) = 4$$

2.

$$f'(0) = \lim_{h\to 0} \frac{h^3}{h}$$
$$= \lim_{h\to 0} h^2 = 0$$

3.

$$f'(a) = \lim_{h\to 0} \frac{m(a+h) + b - (ma+b)}{h}$$
$$= \lim_{h\to 0} \frac{mh}{h} = m$$

Radicals / Rationals

4.

$$f'(1) = \lim_{h\to 0} \frac{\sqrt{1+h} - 1}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{1+h}+1)}$$
$$= \frac{1}{2}$$

5.

$$f'(3) = \lim_{h\to 0} \frac{\frac{1}{3+h} - \frac{1}{3}}{h}$$
$$= \lim_{h\to 0} \frac{3 - (3+h)}{h \cdot 3(3+h)}$$
$$= \lim_{h\to 0} \frac{-h}{h \cdot 3(3+h)} = -\tfrac{1}{9}$$

6.

$$f'(4) = \lim_{h\to 0} \frac{\sqrt{5+h} - \sqrt{5}}{h} \cdot \frac{\sqrt{5+h}+\sqrt{5}}{\sqrt{5+h}+\sqrt{5}}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{5+h}+\sqrt{5})}$$
$$= \tfrac{1}{2\sqrt{5}}$$

Tangent Lines at a Point

7. $y = 6x - x^2$ *at* $(2, 8)$. Check: $6(2) - 2^2 = 12 - 4 = 8$. Slope at $x = 2$ by the limit definition:

$$m = \lim_{h\to 0} \frac{[6(2+h) - (2+h)^2] - [6\cdot 2 - 2^2]}{h}$$
$$= \lim_{h\to 0} \frac{12 + 6h - (4 + 4h + h^2) - (12-4)}{h}$$
$$= \lim_{h\to 0} (2-h) = 2.$$

Tangent line: $y - 8 = 2(x-2)$, i.e. $y = 2x + 4$.

8. $y = 1 + 5x^2$.

(a) Slope at $x = a$:

$$m(a) = \lim_{h\to 0} \frac{[1 + 5(a+h)^2] - (1 + 5a^2)}{h}$$
$$= \lim_{h\to 0} \frac{5(2ah + h^2)}{h}$$
$$= \lim_{h\to 0} (10a + 5h) = 10a.$$

(b) At $(1, 6)$: slope $m = 10$. Tangent line $y - 6 = 10(x-1)$, i.e. $y = 10x - 4$.

9. $y = \sqrt{x+4}$ *at* $x = 5$. Slope via the limit definition (rationalize):

$$m = \lim_{h\to 0} \frac{\sqrt{(5+h)+4} - \sqrt{5+4}}{h} \cdot \frac{\sqrt{9+h}+\sqrt{9}}{\sqrt{9+h}+\sqrt{9}}$$
$$= \lim_{h\to 0} \frac{h}{h\,[\sqrt{9+h}+3]}$$
$$= \frac{1}{6}.$$

Point is $(5, 3)$. Tangent: $y - 3 = \frac{1}{6}(x-5)$, i.e. $y = \frac{1}{6}x + \frac{13}{6}$.

10. The tangent to $y = f(x)$ at $(4, -1)$ passes through $(0, -5)$. The slope of that line is $m = \dfrac{-1 - (-5)}{4 - 0} = 1$. Hence $f(4) = -1$ and $f'(4) = 1$.

11. The tangent to $y = f(x)$ at $(1, 2)$ passes through $(-3, -6)$. The slope is $m = \dfrac{2 - (-6)}{1 - (-3)} = 2$. Hence $f(1) = 2$ and $f'(1) = 2$.

Recognize "Derivative from a Limit"

12. $f(x) = 2x^2 - 5,\ a = 3$.

13. $f(x) = \sqrt{x},\ a = 9$.

14. $f(x) = \ln x,\ a = 1$.