5.5 The Substitution Rule

Theorem (The Substitution Rule)**.** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du.$$

When we let $u = g(x)$, we have

$$du = g'(x)\, dx.$$

The symbols du and dx are called *differentials* and help us track how small changes in u relate to small changes in x; see Section 3.10 for more details.

Proof. Let $u = g(x)$ and let F be an antiderivative of f.

$$\int f(g(x)) \cdot g'(x)\, dx = \int F'(g(x)) \cdot g'(x)\, dx$$

Chain Rule

$$= F(g(x)) + C$$

$$= F(u) + C$$

$$= \int F'(u)\, du$$

$$= \int f(u)\, du$$

□

Example. Find $\int x^3 \cos(x^4 + 2)\, dx$.

Let $u = x^4 + 2$. Then $du = 4x^3 \cdot dx \Rightarrow x^3 dx = \frac{1}{4} du$

$$\int x^3 \cos(x^4+2)\, dx = \int \cos(u) \cdot \frac{1}{4} du$$

$$= \frac{1}{4} \int \cos(u)\, du$$

$$= \frac{1}{4} \sin(u) + C$$

$$= \frac{1}{4} \sin(x^4+2) + C$$

Example. Evaluate $\int \sqrt{2x+1}\, dx$.

Let $u = 2x+1$. Then $du = 2 \cdot dx \Rightarrow \frac{1}{2} du = dx$

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u} \cdot \frac{1}{2} du$$

$$= \frac{1}{2} \int u^{1/2}\, du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C$$

$$= \frac{1}{3} u^{3/2} + C$$

$$= \frac{1}{3} (2x+1)^{3/2} + C$$

Example. Find $\int \frac{x}{\sqrt{1-4x^2}}\,dx$.

Let $u = 1-4x^2$. Then $du = -8x \cdot dx \Rightarrow -\frac{1}{8}\,du = x \cdot dx$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{1}{\sqrt{u}} \cdot -\frac{1}{8}\,du = -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8} \cdot \frac{u^{1/2}}{1/2} + C$$

$$= -\frac{1}{4}u^{1/2} + C$$

$$= -\frac{1}{4}(1-4x^2)^{1/2} + C$$

Example. Calculate $\int e^{5x}\,dx$.

Let $u = 5x$. Then $du = 5 \cdot dx \Rightarrow \frac{1}{5}\,du = dx$

$$\int e^{5x}\,dx = \int e^u \cdot \frac{1}{5}\,du = \frac{1}{5}\int e^u\,du$$

$$= \frac{1}{5}e^u + C$$

$$= \frac{1}{5}e^{5x} + C$$

Example. Calculate $\int x^5\sqrt{1+x^2}\,dx$. $= \int x^4 \cdot x \cdot \sqrt{1+x^2}\,dx$

Let $u = 1 + x^2$. Then $du = 2x \cdot dx \Rightarrow \frac{1}{2}du = x \cdot dx$

Have an extra x^4 factor: $x^4 = (x^2)^2 = (u-1)^2$.

$$\int x^5 \sqrt{1+x^2}\,dx = \int \sqrt{u} \cdot (u-1)^2 \cdot \tfrac{1}{2}\,du$$

$$= \tfrac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\,du$$

$$= \tfrac{1}{2}\int u^{5/2} - 2u^{3/2} + u^{1/2}\,du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2\cdot\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C$$

$$= \tfrac{1}{7}(1+x^2)^{7/2} - \tfrac{2}{5}(1+x^2)^{5/2} + \tfrac{1}{3}(1+x^2)^{3/2} + C$$

Example. Calculate $\int \tan x\,dx$.

- Write $\int \tan x\,dx$ as $\int \frac{\sin x}{\cos x}\,dx$
- Let $u = \cos x \Rightarrow du = -\sin(x)\,dx$

  $\Rightarrow -du = \sin(x)\,dx$

- Hence

$$\int \tan x\,dx = \int \frac{1}{\cos x}\cdot \sin x\,dx = \int \frac{1}{u}\cdot -du$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + c$$

$$= -\ln(|\cos x|) + C$$

Theorem (The Substitution Rule for Definite Integrals)**.** If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Example. Evaluate $\int_0^4 \sqrt{2x+1}\, dx$.

Let $u = 2x+1 \Rightarrow du = 2 \cdot dx \Rightarrow dx = \frac{1}{2} du$

Change bounds:

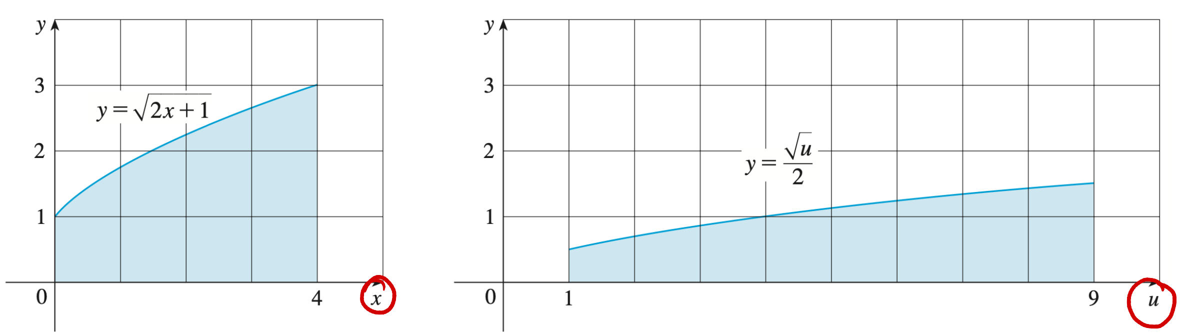
when $x = 0$: $u = 2 \cdot 0 + 1 = 1$

When $x = 4$: $u = 2 \cdot 4 + 1 = 9$

$$\int_{x=0}^{x=4} \sqrt{2x+1}\, dx = \int_{u=1}^{u=9} \sqrt{u} \cdot \frac{1}{2} du = \left[\frac{1}{2} \cdot \frac{u^{3/2}}{3/2} \right]_{u=1}^{u=9}$$

$$= \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right)$$

$$= \frac{26}{3}$$



The shaded areas are the same.

Example. Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.

Let $u = 3-5x$. Then $du = -5\,dx \Rightarrow dx = -\frac{1}{5}du$

Change the bounds:

When $x=1$: $u = 3-5\cdot 1 = -2$

When $x=2$: $u = 3-5\cdot 2 = -7$

$$\int_{x=1}^{x=2} \frac{dx}{(3-5x)^2} = \int_{u=-2}^{u=-7} \frac{1}{u^2} \cdot -\frac{1}{5}du = -\frac{1}{5}\left[-\frac{1}{u}\right]_{u=-2}^{u=-7}$$

$$= -\frac{1}{5}\left[\frac{1}{7} - \frac{1}{2}\right]$$

$$= \boxed{\frac{1}{14}}$$

Example. Calculate $\int_1^e \frac{\ln x}{x}\,dx$.

Let $u = \ln(x) \Rightarrow du = \frac{1}{x}dx$

When $x=1$: $u = \ln(1) = 0$

When $x=e$: $u = \ln(e) = 1$

$$\int_{x=1}^{x=e} \ln(x)\cdot \frac{1}{x}dx = \int_{u=0}^{u=1} u \cdot du = \left[\frac{u^2}{2}\right]_{u=0}^{u=1}$$

$$= \frac{1}{2}$$