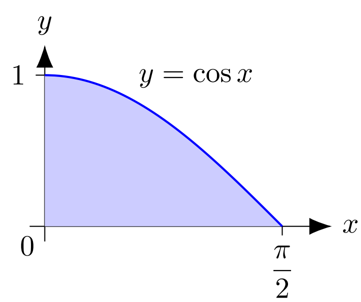
5.4 FTC Part 2, Indefinite Integrals (Solutions)

1. Evaluate $\displaystyle\int_1^3 e^x\,dx$.

Solution. Since $\dfrac{d}{dx}(e^x) = e^x$, an antiderivative of e^x is $F(x) = e^x$. By the Evaluation Theorem,

$$\int_1^3 e^x\,dx = \big[e^x\big]_1^3 = e^3 - e.$$

2. Find the area under the cosine curve from 0 to $\pi/2$.



Solution. The area under $y = \cos x$ from 0 to $\pi/2$ is

$$\int_0^{\pi/2} \cos x\,dx = \big[\sin x\big]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1.$$

Since $\cos x \geq 0$ on $[0, \pi/2]$, this integral equals the (positive) area.

$$\boxed{\text{Area} = 1}$$

3. What is wrong with the following calculation?

$$\int_{-1}^3 \frac{1}{x^2}\,dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^3 = -\frac{1}{3} - 1 = -\frac{4}{3}.$$

Solution. The integrand $f(x) = \dfrac{1}{x^2}$ is *not* continuous on $[-1, 3]$ because it is undefined at $x = 0$. Therefore we cannot directly apply the Evaluation Theorem on $[-1, 3]$.

4. Find the indefinite integral $\int (10x^4 - 2\sec^2 x)\,dx$.

Solution. Integrate term by term:

$$\int (10x^4 - 2\sec^2 x)\,dx = 2x^5 - 2\tan x + C$$

5. Evaluate $\int_0^3 (x^3 - 6x)\,dx$.

Solution. An antiderivative is

$$\int (x^3 - 6x)\,dx = \frac{x^4}{4} - 3x^2 + C.$$

Therefore

$$\int_0^3 (x^3 - 6x)\,dx = \left[\frac{x^4}{4} - 3x^2\right]_0^3 = \left(\frac{3^4}{4} - 3\cdot 3^2\right) - \left(\frac{0^4}{4} - 3\cdot 0^2\right) = \frac{81}{4} - 27 = -\frac{27}{4}.$$

The integral is negative because the graph of $y = x^3 - 6x$ lies mostly below the x-axis on $[0, 3]$.

6. Find $\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx$ and interpret the result in terms of areas.

Solution. First find an antiderivative:

$$\int \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx = \frac{x^4}{2} - 3x^2 + 3\arctan x + C.$$

Evaluate from 0 to 2:

$$\begin{aligned}\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right) dx &= \left[\frac{x^4}{2} - 3x^2 + 3\arctan x\right]_0^2 \\ &= \left(\frac{16}{2} - 12 + 3\arctan 2\right) - (0 - 0 + 0). \\ &= -4 + 3\arctan 2\end{aligned}$$

This is the *net signed area* between $y = 2x^3 - 6x + \frac{3}{x^2+1}$ and the x-axis on $[0, 2]$.

7. Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt$.

Solution. First simplify the integrand:

$$\frac{2t^2 + t^2\sqrt{t} - 1}{t^2} = 2 + \sqrt{t} - \frac{1}{t^2} = 2 + t^{1/2} - t^{-2}.$$

Now integrate term by term:

$$\int (2 + t^{1/2} - t^{-2})\, dt = 2t + \frac{2}{3}t^{3/2} + \frac{1}{t} + C.$$

Evaluate from 1 to 9:

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\, dt = \left[2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\right]_1^9 = \left(18 + \frac{2}{3}9^{3/2} + \frac{1}{9}\right) - \left(2 + \frac{2}{3} + 1\right) = \frac{292}{9}.$$

8. A particle moves along a line so that its velocity (m/s) at time t is $v(t) = t^2 - t - 6$.

(a) Find the displacement of the particle during the time period $1 \le t \le 4$.

(b) Find the distance traveled during this time period.

Solution.

(a) Displacement is the net change in position:

$$\text{displacement} = \int_1^4 v(t)\, dt = \int_1^4 (t^2 - t - 6)\, dt = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = -\frac{9}{2}.$$

The negative sign means the particle ends up 4.5m to the *left* of its position at $t = 1$.

(b) Distance is the integral of the *speed* $|v(t)|$:

$$\text{distance} = \int_1^4 |v(t)|\, dt.$$

Find where $v(t)$ changes sign on $[1, 4]$:

$$v(t) = t^2 - t - 6 = (t - 3)(t + 2).$$

The roots are $t = -2$ and $t = 3$. Using a sign chart, on $[1, 4]$ we have

$$v(t) < 0 \text{ for } t \text{ in } [1, 3), \qquad v(t) > 0 \text{ for } t \text{ in } (3, 4].$$

Thus

$$\int_1^4 |v(t)|\, dt = \int_1^3 -v(t)\, dt + \int_3^4 v(t)\, dt = \frac{22}{3} + \frac{17}{6} = \frac{61}{6} \text{ meters}.$$