5.4 FTC Part 2, Indefinite Integrals

Theorem (Evaluation Theorem / FTC Part II)**.** Let f be continuous on $[a, b]$ and let F be any antiderivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Step 1. Partition the interval. Let n be a positive integer. Divide $[a, b]$ into n equal subintervals with endpoints

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

each of length

$$\Delta x = \underline{\frac{b-a}{n}}.$$

Step 2. Write $F(b) - F(a)$ as a telescoping sum.

$$F(b) - F(a) = F(x_n) - F(x_0)$$

$$= [F(x_n) - \cancel{F(x_{n-1})}] + [\cancel{F(x_{n-1})} - \cancel{F(x_{n-2})}] + \cdots + [\cancel{F(x_1)} - F(x_0)]$$

$$= \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$

Step 3. Apply the Mean Value Theorem on each subinterval. On each subinterval $[x_{i-1}, x_i]$, the function F is continuous and differentiable, so by the Mean Value Theorem there exists a point x_i^* in $[x_{i-1}, x_i]$with

$$F'(x_i^*) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

In other words,

$$F(x_i) - F(x_{i-1}) = F'(x_i^*) \cdot (x_i - x_{i-1}) = F'(x_i^*) \cdot \Delta x$$

Therefore,

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} F'(x_i^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Step 4. Recognize a Riemann sum. Since f is continuous on $[a,b]$, the limit of these Riemann sums exists and equals the definite integral:

$$F(b)-F(a) = \lim_{n\to\infty} F(b) - F(a)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} f(x_i^*)\Delta x$$

[def. of integral] $$= \int_a^b f(x)\,dx$$

Notation: We often write this as

$$\int_a^b f(x)\,dx = \big[F(x)\big]_a^b = F(b)-F(a).$$

Example. Evaluate $\int_0^1 x^2\,dx$.

An antiderivative is $F(x) = \frac{1}{3}x^3 + C$ [take $C=0$]

$$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \left[\tfrac{1}{3}\cdot 1^3\right] - \left[\tfrac{1}{3}\cdot 0^3\right]$$

$$= \frac{1}{3} - 0$$

$$= \frac{1}{3}$$

[Compare this to what we did in §5.1]

Definition (Indefinite Integral)**.** Let f be a function defined on an interval I. A function F is called an *antiderivative* of f on I if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

The "most general" antiderivative

The *indefinite integral* of f with respect to x is the family of all antiderivatives of f and is written

$$\int f(x)\,dx = F(x) + C,$$

where $F'(x) = f(x)$ on I and C is an arbitrary constant.

Indefinite Integral Formulas

$$\int c\,f(x)\,dx = c\int f(x)\,dx$$

$$\int \big[f(x) + g(x)\big]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int b^x\,dx = \frac{b^x}{\ln b} + C, \quad b > 0,\ b \neq 1$$

$$\int e^x\,dx = e^x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc x\,\cot x\,dx = -\csc x + C$$

$$\int \sec x\,\tan x\,dx = \sec x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{1}{x^2+1}\,dx = \arctan x + C$$