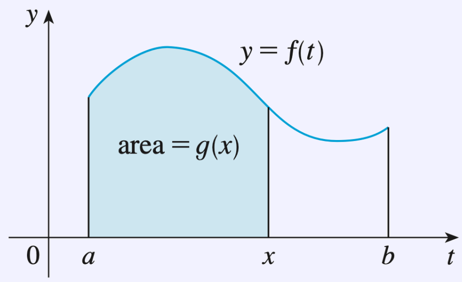
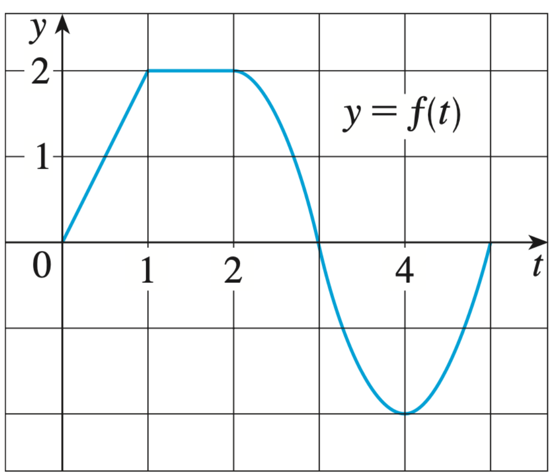
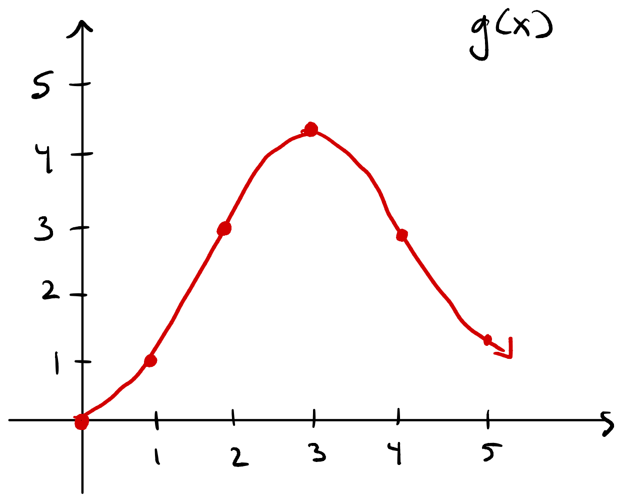
5.3 The Fundamental Theorem of Calculus

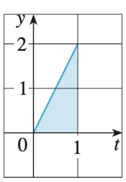
Definition. An **accumulation function** function is a function of the form $g(x) = \int_a^x f(t)dt$. It represents the area under a curve $f(t)$ from a to x (i.e. the "area so far").



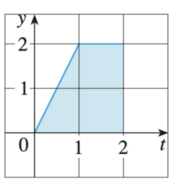
Example. If f is the function whose graph is shown below and $g(x) = \int_0^x f(t)dt$, find the values of $g(0), g(1), g(2), g(3), g(4)$, and $g(5)$. Then sketch a rough graph of g.



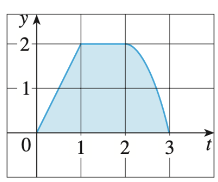




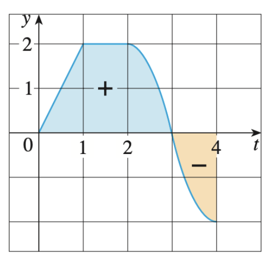
$g(1) = 1$



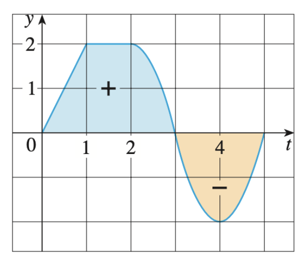
$g(2) = 3$



$g(3) \approx 4.3$



$g(4) \approx 3$



$g(5) \approx 1.7$

✱ Note: When $g(x)$ is increasing, $f(x) > 0$.

When $g(x)$ is decreasing, $f(x) < 0$.

Example. If $g(x) = \int_a^x f(t)dt$, where $a = 1$ and $f(t) = t^2$, find a formula for $g(x)$ and $g'(x)$.

$$g(x) = \int_1^x t^2\,dt = \left[\frac{1}{3}t^3\right]_1^x = \frac{1}{3}x^3 - \frac{1}{3}\cdot 1^3 = \frac{1}{3}x^3 - \frac{1}{3}$$

$$g'(x) = x^2$$

Note: $g'(x) = f(x)$

Theorem (Fundamental Theorem of Calculus, Part 1). If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt, \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

The instantaneous change in the "accumulation" of $f(t)$ at a point is the value of the function at that point.

Example. Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}dt$.

Since $f(t) = \sqrt{1+t^2}$ is continuous, $g'(x) = \sqrt{1+x^2}$ by FTC 1

Example. Find $\frac{d}{dx}\int_1^{x^4} \sec t\,dt$.

⚠ Must use chain rule when the bounds are functions of x.

Let $y = \int_1^u \sec t\,dt$ and $u = x^4$

$$\frac{dy}{dx} = \underbrace{\frac{dy}{du}}_{\text{FTC 1}}\cdot\frac{du}{dx} = \sec(u)\cdot 4x^3 = \sec(x^4)\cdot 4x^3$$