5.2 The Definite Integral

Definition. Let f be a function defined on the closed interval $[a, b]$. Divide $[a, b]$ into n equal pieces of width

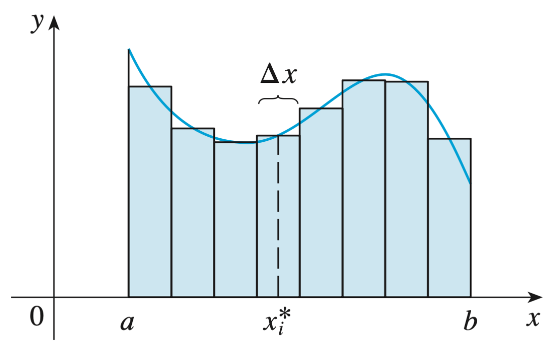
$$\Delta x = \frac{b-a}{n},$$

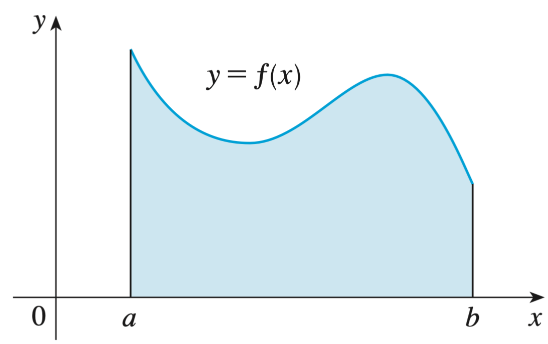
with endpoints $x_0 = a, x_1, \ldots, x_n = b$. In each subinterval $[x_{i-1}, x_i]$, choose any *sample point* x_i^* (for example, a left endpoint, right endpoint, or midpoint). Then the **definite integral** of f from a to b is

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

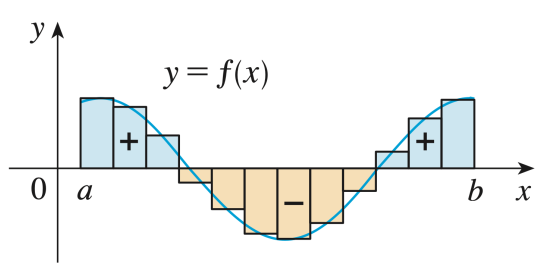
provided this limit exists and gives the same value for *all* choices of sample points $\{x_i^*\}$. When this limit exists, we say that f is *integrable* on $[a, b]$.

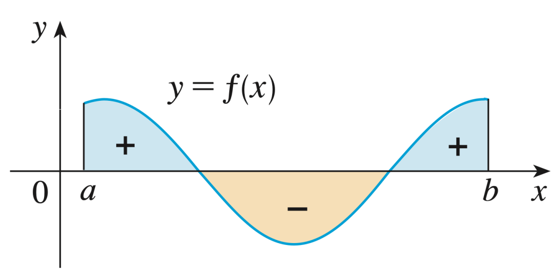
Question. If f is positive on the interval $[a, b]$, how can we interpret the definite integral?





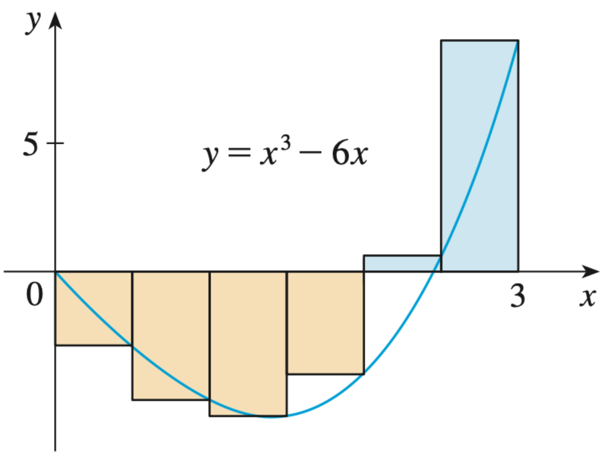
Question. If f takes on both positive and negative values on the interval $[a, b]$, how can we interpret the definite integral?



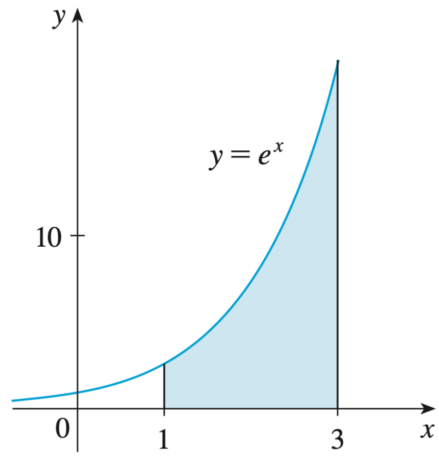


Example. Express $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$ as an integral on the interval $[0, \pi]$.

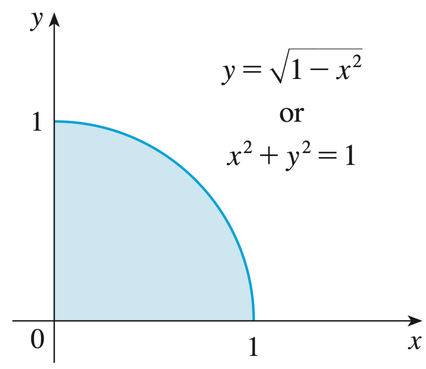
Example. Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.



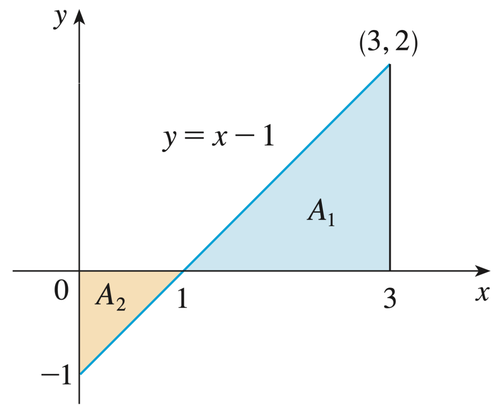
Example. Set up an expression for $\int_1^3 e^x dx$ as a limit of sums.



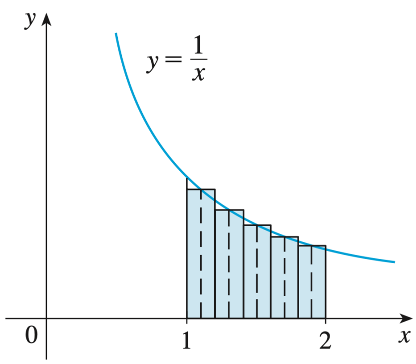
Example. Evaluate $\int_0^1 \sqrt{1-x^2}\, dx$ by interpreting it in terms of areas.



Example. Evaluate $\int_0^3 (x-1)\,dx$ by interpreting it in terms of areas.



Example. Use M_5 to approximate $\int_1^2 \frac{1}{x}\,dx$



Theorem (Properties of the Definite Integral)**.** If f and g are continuous functions, then

1. $\displaystyle\int_b^a f(x)\,dx \;=\; -\int_a^b f(x)\,dx.$

2. $\displaystyle\int_a^a f(x)\,dx \;=\; 0.$

3. $\displaystyle\int_a^b c\,dx \;=\; c\,(b-a).$

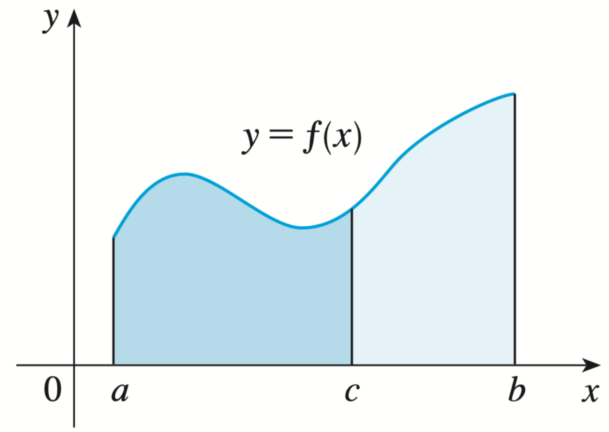
4. $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,dx \;=\; \int_a^b f(x)\,dx \;+\; \int_a^b g(x)\,dx.$

5. $\displaystyle\int_a^b c\,f(x)\,dx \;=\; c\int_a^b f(x)\,dx.$

6. $\displaystyle\int_a^b \big(f(x)-g(x)\big)\,dx \;=\; \int_a^b f(x)\,dx \;-\; \int_a^b g(x)\,dx.$

Example. Use the properties of integrals to evaluate $\displaystyle\int_0^1 (4+3x^2)dx$.

Question. How do we combine integrals of the same function over adjacent intervals?

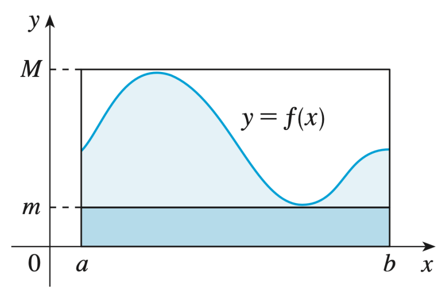


Example. If it is known that $\int_0^{10} f(x)dx = 17$ and $\int_0^{8} f(x)dx = 12$, find $\int_8^{10} f(x)dx$.

Theorem (Comparison Properties of the Integral)**.** The following properties are true only if $a \leq b$.

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)dx \geq 0$.
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $\displaystyle m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

Proof.



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Example. Estimate $\displaystyle\int_0^1 e^{-x^2}dx$.