5.2 The Definite Integral (Activity)

Step 1: Cut the interval. Split $[a, b]$ into n equal pieces:

$$\Delta x = \frac{b-a}{n}, \qquad x_i = a + i\,\Delta x \quad (i = 0, 1, \ldots, n).$$

Step 2: Pick one point in each small interval $[x_{i-1}, x_i]$. Common choices:

$$\text{Left: } x_{i-1} \qquad \text{Right: } x_i \qquad \text{Midpoint: } \frac{x_{i-1} + x_i}{2}.$$

Step 3: Add up rectangle areas.

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x, \qquad R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x, \qquad M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x$$

Definite Integral: The *net signed area* under f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x, \quad \text{where } x_i^* \text{ in } [x_{i-1}, x_i] \text{ can be any sample point.}$$

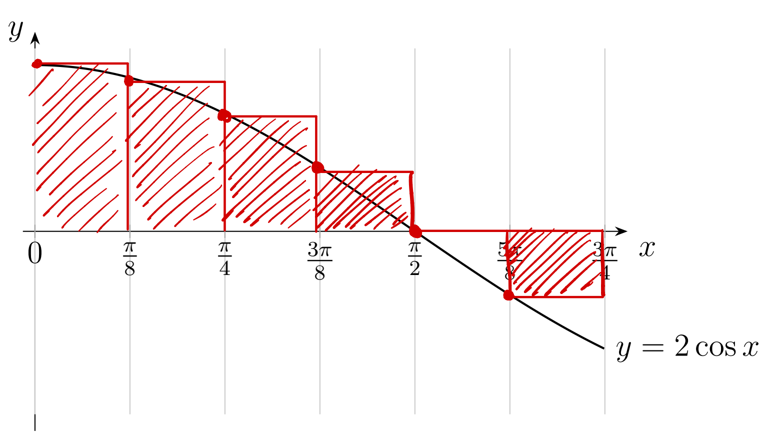
1. For $f(x) = 2\cos x$ on $[0, \frac{3\pi}{4}]$ with $n = 6$: $\Delta x = \frac{3\pi/4 - 0}{6} = \frac{\pi}{8}$

   (a) Compute L_6 to six decimal places.

   $L_6 = 2\cos(0)\cdot\frac{\pi}{8} + 2\cos(\frac{\pi}{8})\cdot\frac{\pi}{8} + 2\cos(\frac{\pi}{4})\cdot\frac{\pi}{8} + 2\cos(\frac{3\pi}{8})\cdot\frac{\pi}{8} + 2\cos(\frac{\pi}{2})\cdot\frac{\pi}{8} + 2\cos(\frac{5\pi}{8})\cdot\frac{\pi}{8}$

   ≈ 2.066372

   (b) Sketch the rectangles you used to compute L_6 (left endpoints).



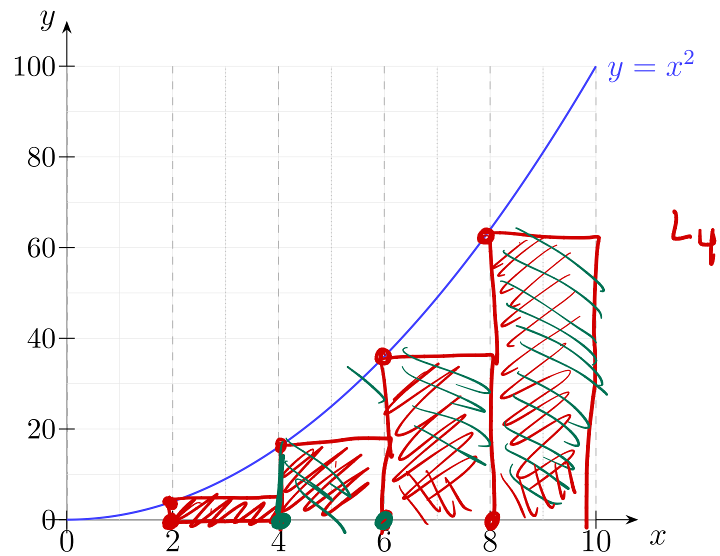
2. Estimate $\int_2^{10} x^2\,dx$ by computing L_4 and R_4. $\Delta x = \frac{10-2}{4} = 2$ ← Base of each rectangle

$L_4 = 2^2 \cdot 2 + 4^2 \cdot 2 + 6^2 \cdot 2 + 8^2 \cdot 2$

$= 240$

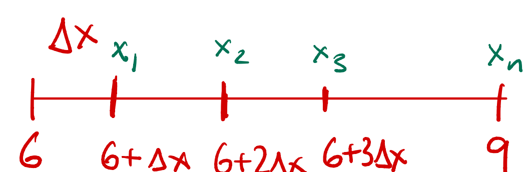
$R_4 = 4^2 \cdot 2 + 6^2 \cdot 2 + 8^2 \cdot 2 + 10^2 \cdot 2$

$= 432$



3. On $[2, 6]$, write $\lim_{n\to\infty} \sum_{i=1}^{n} (x_i^3 + x_i)\,\Delta x$ as a definite integral.

This is the definition of $\int_2^6 x^3 + x\,dx$, where the sample points are right endpoints.



4. Write $\int_6^9 \left(x^2 + \frac{1}{x}\right) dx$ as a limit of Riemann sums using *right* endpoints.

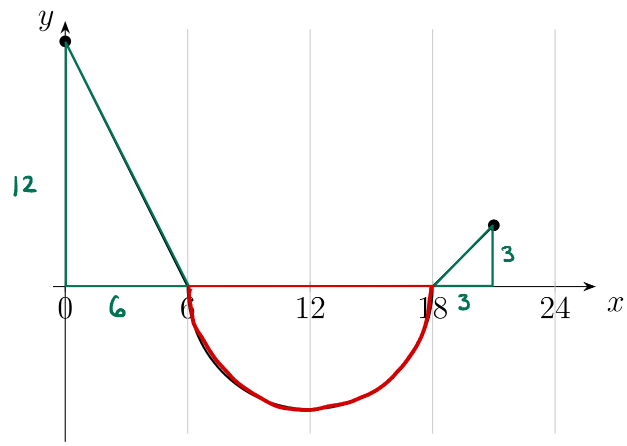
$$\int_6^9 x^2 + \frac{1}{x}\,dx = \lim_{\substack{n\to\infty \\ \text{\# Rect} \to \infty}} \sum_{i=1}^{n} \underbrace{\left(x_i^2 + \frac{1}{x_i}\right)}_{\text{height of rect}} \cdot \underbrace{\Delta x}_{\text{base}}$$

$\Delta x = \frac{9-6}{n} = \frac{3}{n}$ and the right endpoints are $x_i = 6 + i\Delta x = 6 + \frac{3i}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(6 + \frac{3i}{n}\right)^2 + \frac{1}{6 + \frac{3i}{n}}\right] \cdot \frac{3}{n}$$

5. Below, g is piecewise:

- $g(x) = 12 - 2x$ on $[0, 6]$,
- a semicircle of radius 6 *below* the x-axis on $[6, 18]$,
- linear from $(18, 0)$ to $(21, 3)$.



Evaluate:

(a) $\int_0^6 g(x)\,dx$ 36

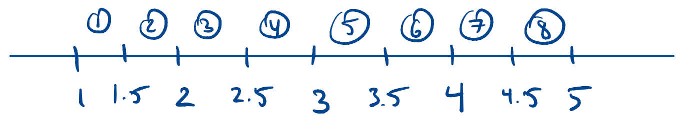
(b) $\int_6^{18} g(x)\,dx$ -18π

(c) $\int_0^{21} g(x)\,dx$ $40.5 - 18\pi$

6. Suppose $\int_1^7 f(x)\,dx = 9.2$ and $\int_6^7 f(x)\,dx = 5.6$. Find $\int_1^6 f(x)\,dx$.

$\int_1^6 f(x)\,dx = 9.2 - 5.6 = 3.6$

7. On $[1, 5]$ with $n = 8$, write M_8 for $f(x) = \sqrt{x}$. State Δx and the 8 midpoints.



$\Delta x = \frac{5-1}{8} = 0.5$

Midpoints: 1.25, 1.75, 2.25, 2.75, 3.25, 3.75, 4.25, 4.75

$M_8 = \sqrt{1.25} \cdot 0.5 + \sqrt{1.75} \cdot 0.5 + \ldots + \sqrt{4.75} \cdot 0.5$

8. Express $\int_2^{10} (3x^2 - 4x + 1)\, dx$ as a limit of right-endpoint sums. (Do not evaluate.)

Let $\Delta x = \frac{10-2}{n} = \frac{8}{n}$. Right endpoints are $x_i = 2 + i\Delta x = 2 + \frac{8i}{n}$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \left[3x_i^2 - 4x_i^2 + 1\right] \Delta x = \lim_{n \to \infty} \sum_{i=1}^{\infty} \left[3\left(2 + \frac{8i}{n}\right)^2 - 4\left(2 + \frac{8i}{n}\right) + 1\right] \cdot \frac{8}{n}$$

9. Write $\lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n}$ as a definite integral.

$\Delta x = \frac{1}{n}$, right endpoints $x_i = \frac{i}{n}$

$\int_0^1 x^3\, dx$