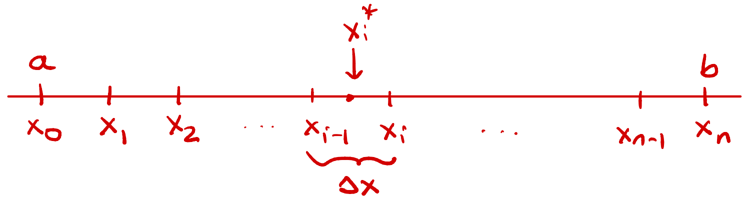


5.2 The Definite Integral

Definition. Let f be a function defined on the closed interval $[a, b]$. Divide $[a, b]$ into n equal pieces of width

$$\Delta x = \frac{b-a}{n},$$

← base of each rectangle

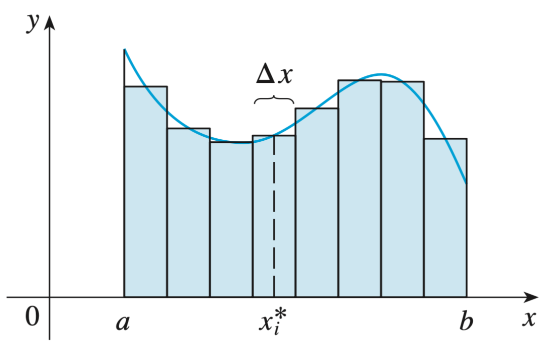
with endpoints $x_0 = a, x_1, \ldots, x_n = b$. In each subinterval $[x_{i-1}, x_i]$, choose any *sample point* x_i^* (for example, a left endpoint, right endpoint, or midpoint). Then the **definite integral** of f from a to b is

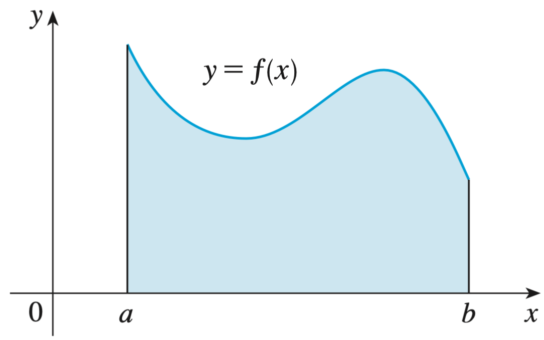
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

height of each rectangle
base of each rectangle
sum all rectangles
limit as # rectangles → ∞

provided this limit exists and gives the same value for *all* choices of sample points $\{x_i^*\}$. When this limit exists, we say that f is *integrable* on $[a, b]$.

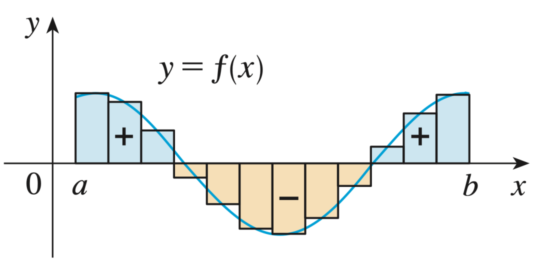
Question. If f is positive on the interval $[a, b]$, how can we interpret the definite integral?

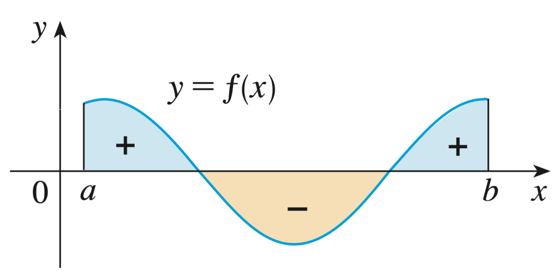




$\int_a^b f(x)\,dx$ is the area under the curve

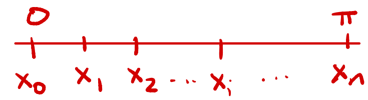
Question. If f takes on both positive and negative values on the interval $[a, b]$, how can we interpret the definite integral?





$\int_a^b f(x)\,dx$ is the <u>net</u> area under the curve.

Example. Express $\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x$ as an integral on the interval $[0, \pi]$.



Let $f(x) = x^3 + x\sin(x)$ and $a = 0$, $b = \pi$

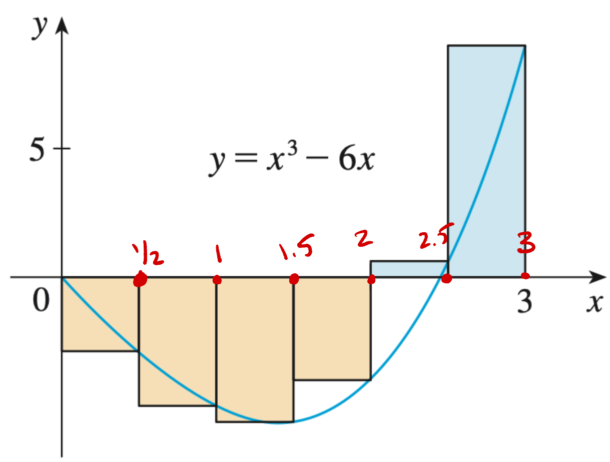
$$\lim_{n\to\infty}\sum_{i=1}^{n}(x_i^3 + x_i \sin x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \int_0^{\pi} f(x)\,dx$$

Definition of definite integral where our sample points are the right endpoints.

$$= \int_0^{\pi} x^3 + \sin x \; dx$$

Example. Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.



$$\Delta x = \frac{3-0}{6} = \frac{1}{2}$$

base of each rectangle

Right endpoints:

$x_1 = 1/2$, $x_2 = 1$, $x_3 = 1.5$

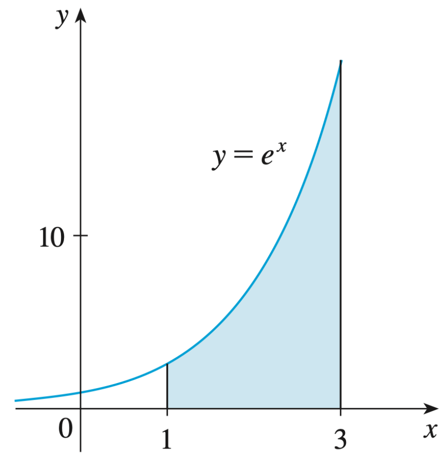
$x_4 = 2$, $x_5 = 2.5$, $x_6 = 3$

$$R_6 = f(0.5)\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(1.5)\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + f(2.5)\cdot\tfrac{1}{2} + f(3)\cdot\tfrac{1}{2}$$

$$\approx -3.9375$$

Example. Set up an expression for $\int_1^3 e^x dx$ as a limit of sums.

$f(x) = e^x, \quad a = 1, \; b = 3$



$\Delta x = \frac{3-1}{n} = \frac{2}{n}$

Sample points ?

$x_0 = 1, \quad x_1 = 1 + \frac{2}{n}, \quad x_2 = 1 + \frac{4}{n}, \quad x_3 = 1 + \frac{6}{n}, \ldots$

$\boxed{x_i = 1 + \frac{2i}{n}}$ ← Formula for i^{th} right endpoint

Def of integral ↓ ; Sample points = right endpoints ↓

$$\int_1^3 e^x dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$$

the right endpoints

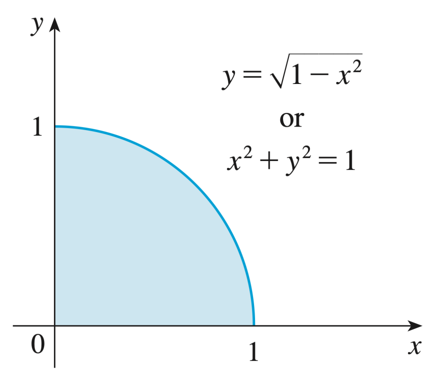
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right) \cdot \frac{2}{n}$$ ← base of rectangle

$$\boxed{= \lim_{n\to\infty} \sum_{i=1}^{n} e^{1+\frac{2i}{n}} \cdot \frac{2}{n}}$$

Example. Evaluate $\int_0^1 \sqrt{1 - x^2}\, dx$ by interpreting it in terms of areas.

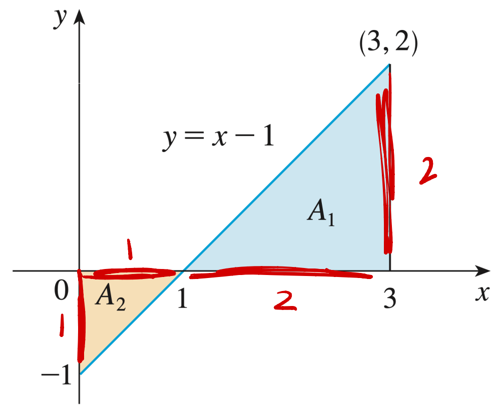
$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{1}{4} \cdot \pi \cdot 1^2 = \frac{\pi}{4}$$

↑ Quarter of a circle with radius 1



Example. Evaluate $\int_0^3 (x-1)\,dx$ by interpreting it in terms of areas.

Compute the integral using the areas of triangles



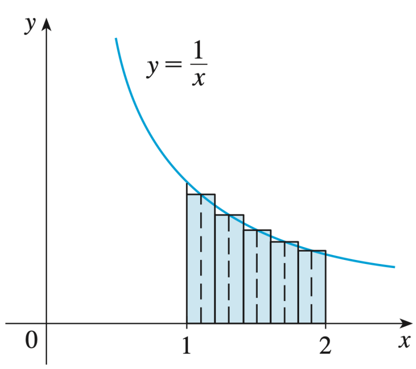
$$\int_0^3 x-1\,dx = A_1 - A_2$$

$$= \tfrac{1}{2}\cdot 2\cdot 2 - \tfrac{1}{2}\cdot 1\cdot 1$$

$$= 2 - \tfrac{1}{2}$$

$$= 1.5$$

Example. Use M_5 to approximate $\int_1^2 \frac{1}{x}\,dx$



$\Delta x = \frac{2-1}{5} = 0.2$

The endpoints are:

$x_0 = 1,\ x_1 = 1.2,\ x_2 = 1.4,\ x_3 = 1.6,\ x_4 = 1.8,\ x_5 = 2$

The midpoints are:

$1.1, 1.3, 1.5, 1.7, 1.9$

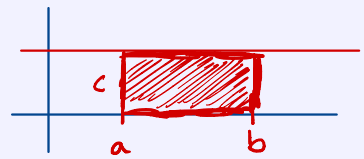
$$\int_1^2 \frac{1}{x}\,dx \approx f(1.1)\cdot\tfrac{1}{5} + f(1.3)\cdot\tfrac{1}{5} + f(1.5)\cdot\tfrac{1}{5} + f(1.7)\cdot\tfrac{1}{5} + f(1.9)\cdot\tfrac{1}{5}$$

$$= \frac{1}{1.1}\cdot\frac{1}{5} + \frac{1}{1.3}\cdot\frac{1}{5} + \frac{1}{1.5}\cdot\frac{1}{5} + \frac{1}{1.7}\cdot\frac{1}{5} + \frac{1}{1.9}\cdot\frac{1}{5}$$

$$\approx 0.691908$$

Theorem (Properties of the Definite Integral)**.** If f and g are continuous functions, then

1. $\displaystyle\int_b^a f(x)\,dx \;=\; -\int_a^b f(x)\,dx.$
2. $\displaystyle\int_a^a f(x)\,dx \;=\; 0.$
3. $\displaystyle\int_a^b c\,dx \;=\; c\,(b-a).$



4. $\displaystyle\int_a^b \big(f(x)+g(x)\big)\,dx \;=\; \int_a^b f(x)\,dx \;+\; \int_a^b g(x)\,dx.$
5. $\displaystyle\int_a^b c\,f(x)\,dx \;=\; c\int_a^b f(x)\,dx.$
6. $\displaystyle\int_a^b \big(f(x)-g(x)\big)\,dx \;=\; \int_a^b f(x)\,dx \;-\; \int_a^b g(x)\,dx.$

Example. Use the properties of integrals to evaluate $\displaystyle\int_0^1 (4+3x^2)dx$.

$$\int_0^1 4+3x^2\,dx = \int_0^1 4\,dx + \int_0^1 3x^2\,dx$$

$$= \int_0^1 4\,dx + 3\underline{\int_0^1 x^2\,dx}$$

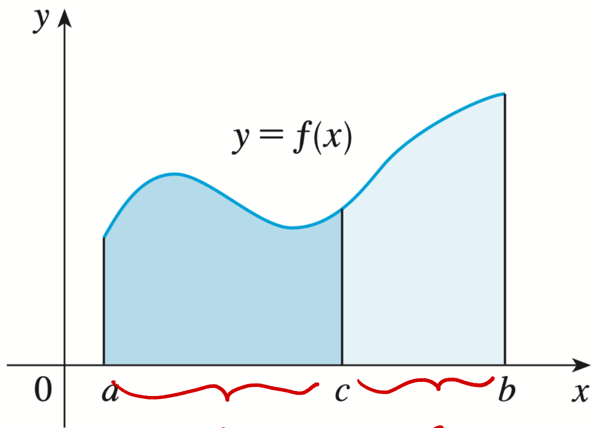
In §5.1, we showed this is 1/3

$$= 4(1-0) + 3\cdot\frac{1}{3}$$

$$= 4+1$$

$$= 5$$

Question. How do we combine integrals of the same function over adjacent intervals?



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example. If it is known that $\displaystyle\int_0^{10} f(x)dx = 17$ and $\displaystyle\int_0^{8} f(x)dx = 12$, find $\displaystyle\int_8^{10} f(x)dx$.

$$\int_0^{10} f(x)\,dx = \int_0^8 f(x)\,dx + \int_8^{10} f(x)\,dx$$

$$17 = 12 + \int_8^{10} f(x)\,dx$$

$$5 = \int_8^{10} f(x)\,dx$$

Theorem (Comparison Properties of the Integral)**.** The following properties are true only if $a \leq b$.

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

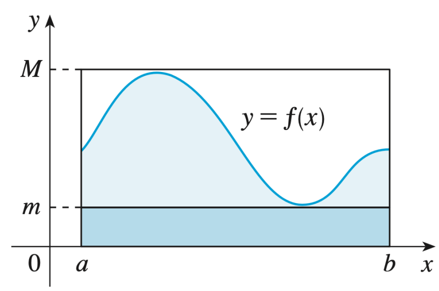
Proof.

Since $m \leq f(x) \leq M$, property #2 says:

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

(The integral $\int_a^b f(x)dx$ is contained within two rectangles)



□

Example. Estimate $\int_0^1 e^{-x^2} dx$.

$f(x) = e^{-x^2}$ is decreasing on $[0,1]$

The max is $M = f(0) = 1$

The min is $m = f(1) = \frac{1}{e}$

$$\frac{1}{e}(1-0) \leq \int_0^1 e^{-x^2}dx \leq 1(1-0)$$

$$\frac{1}{e} \leq \int_0^1 e^{-x^2}dx \leq 1$$

0.367