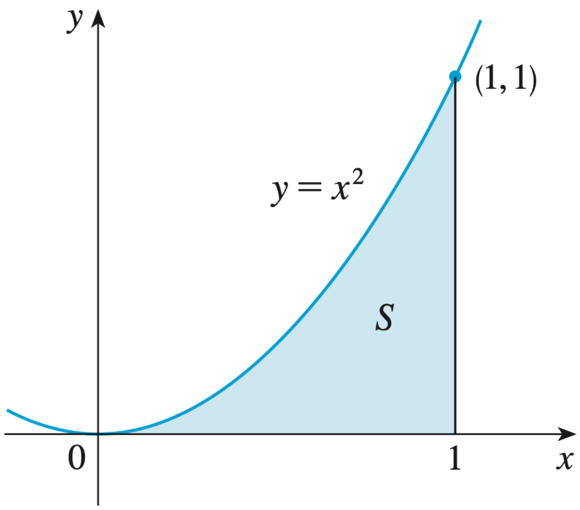
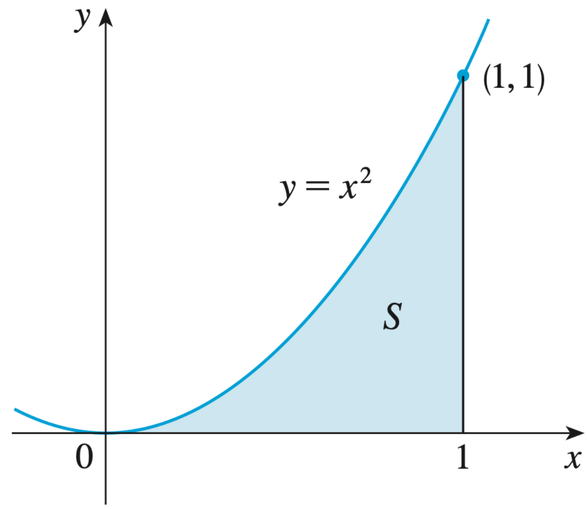
5.1 Areas and Distances

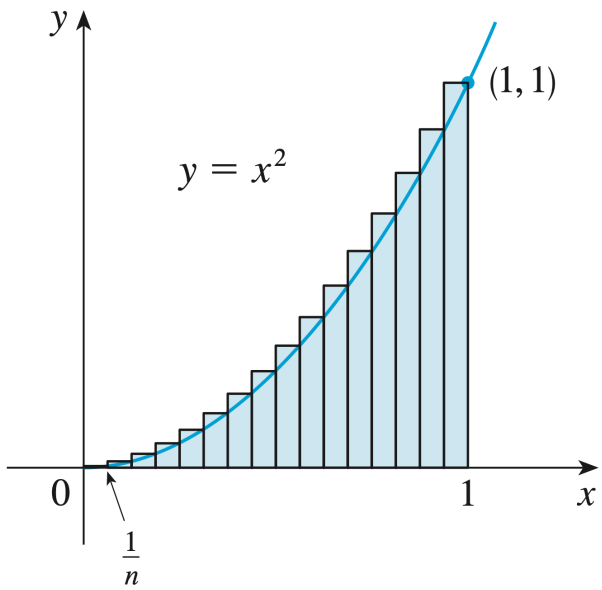
Question. How will we find the area under a curve?

Example. Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

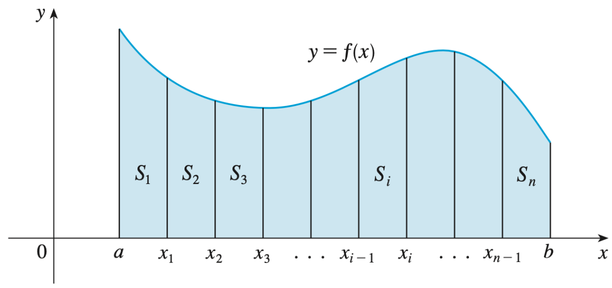


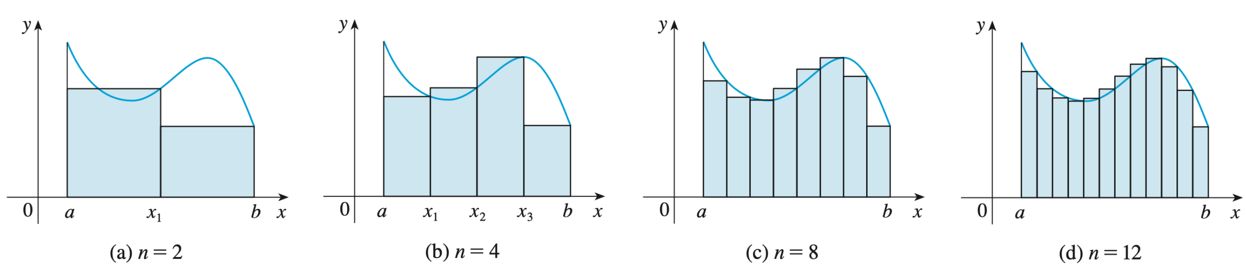

y
(1, 1)
y = x2
S
0
1
x

Example. Let S be the region under the parabola $y = x^2$, as in the previous example. Show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is, $\lim_{n\to\infty} R_n = \frac{1}{3}$.



Example. Apply the idea of the previous example to estimate the area of the more general region below.





Definition. Let f be continuous on an interval $[a, b]$. Divide $[a, b]$ into n equal parts of width $\Delta x = \dfrac{b-a}{n}$ with points $x_0 = a,\ x_1, \ldots, x_n = b$. We have three ways to define the *area* A of the region under $y = f(x)$ from a to b.

1. Using right endpoints:

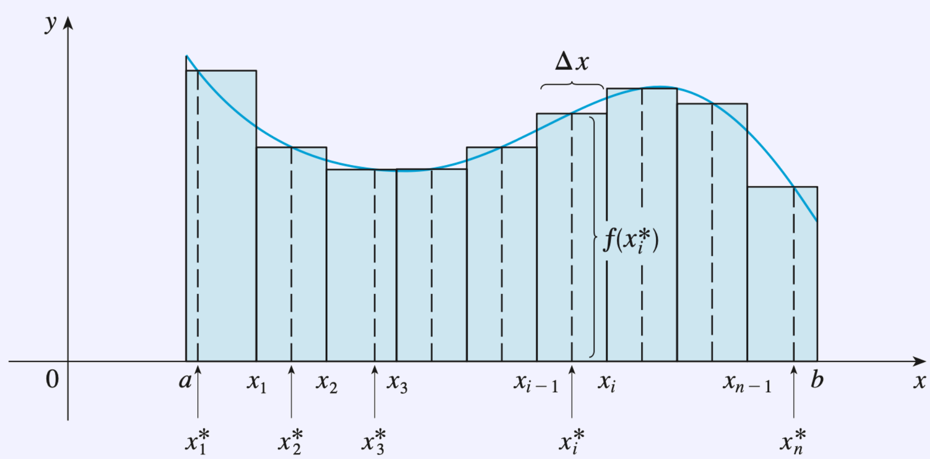
$$R_n = \left[f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x\right], \qquad A = \lim_{n\to\infty} R_n.$$

2. Using left endpoints:

$$L_n = \left[f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x\right], \qquad A = \lim_{n\to\infty} L_n.$$

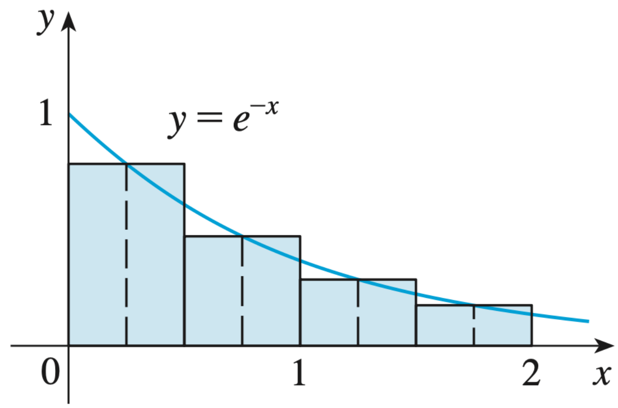
3. General sample points: in each subinterval $[x_{i-1}, x_i]$, choose any point x_i^*. Then,

$$S_n = \left[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x\right], \qquad A = \lim_{n\to\infty} S_n.$$

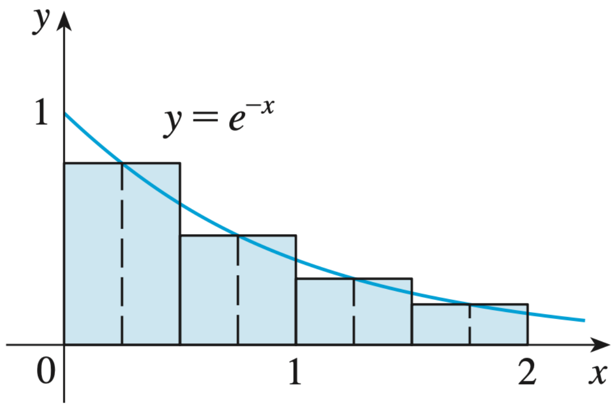


Theorem. For a continuous function f, all three limits above exist and are equal; this common limit is the area A.

Example. Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$. Using **right endpoints**, find an expression for A as a limit.



Example. Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$. Estimate the area of A by taking the sample points to be **midpoints** and using four sub-intervals.



Example. Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28
Velocity (ft/s)							