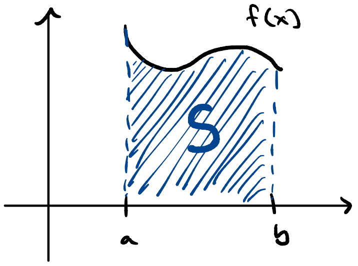
5.1 Areas and Distances

Question. How will we find the area under a curve?

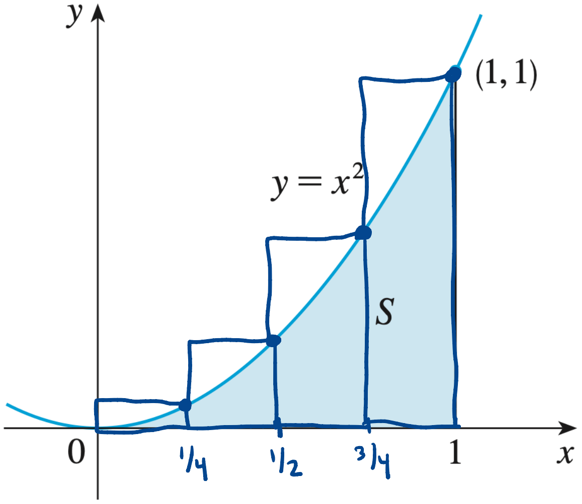


Goal: Find the area of the region S that lies under a curve f(x) between two values a and b

① Approximate this area using rectangles

② Take the limit as # of rectangles → ∞ to get an exact value

Example. Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1.

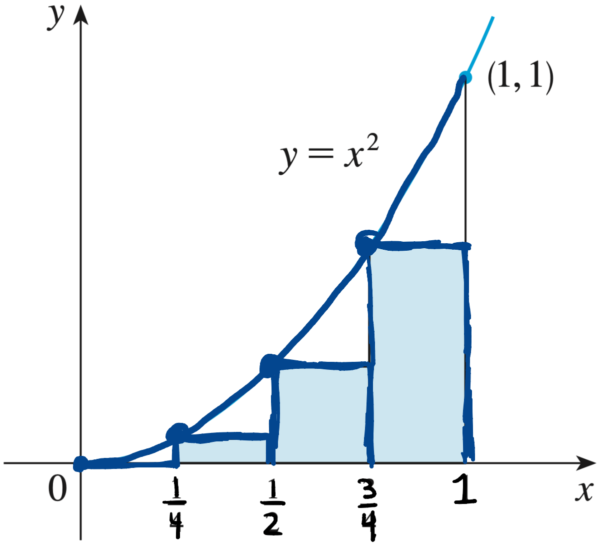


- Divide S into 4 strips
- Approximate each strip by a rectangle whose upper-right point lies on the curve.

$$\text{Area of } S \approx R_4 = \overset{\text{base}}{\left(\tfrac{1}{4}\right)} \cdot \overset{\text{height}}{\left(\tfrac{1}{4}\right)^2} + \left(\tfrac{1}{4}\right)\cdot\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4}\right)\cdot\left(\tfrac{3}{4}\right)^2 + \left(\tfrac{1}{4}\right)\cdot(1)^2 = 0.46875$$

Note: $\boxed{\text{Area of } S \leq 0.46875}$ since S lies within the rectangles

- We could have also used rectangles whose upper-left points are on the curve



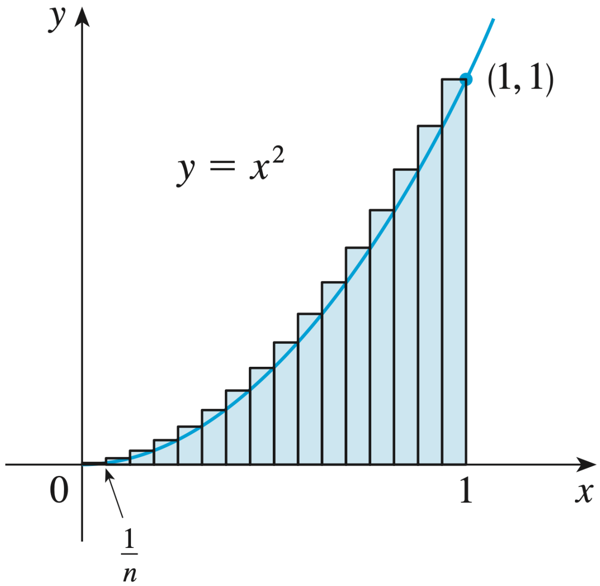
$$\text{Area of } S \approx L_4 = \overset{\text{base}}{\tfrac{1}{4}}\overset{\text{height}}{(0)^2} + \tfrac{1}{4}\left(\tfrac{1}{4}\right)^2 + \tfrac{1}{4}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4}\left(\tfrac{3}{4}\right)^2 = 0.21875$$

Note: $0.21875 \leq \text{Area of } S \leq 0.46875$

Idea: More rectangles = better approximation

As # of rectangles → ∞ both methods approach the same thing!

Example. Let S be the region under the parabola $y = x^2$, as in the previous example. Show that the sum of the areas of the upper approximating rectangles approaches $\frac{1}{3}$, that is, $\lim_{n\to\infty} R_n = \frac{1}{3}$.



If there are n rectangles:

Each has width: $\frac{1}{n}$

The heights are:

$$\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \dots, \left(\frac{n}{n}\right)^2$$

Total Area of the rectangles:

$$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \dots + \underset{\text{base}}{\frac{1}{n}}\underset{\text{height}}{\left(\frac{n}{n}\right)^2}$$

$$= \frac{1}{n}\left(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \frac{3^2}{n^2} + \dots + \frac{n^2}{n^2}\right)$$

$$= \frac{1}{n} \cdot \frac{1}{n^2} \cdot \left(1^2 + 2^2 + 3^2 + \dots + n^2\right)$$

$$= \frac{1}{n^3} \cdot \left(\frac{n(n+1)(2n+1)}{6}\right) \longleftarrow$$

Formula for sum of first n squares

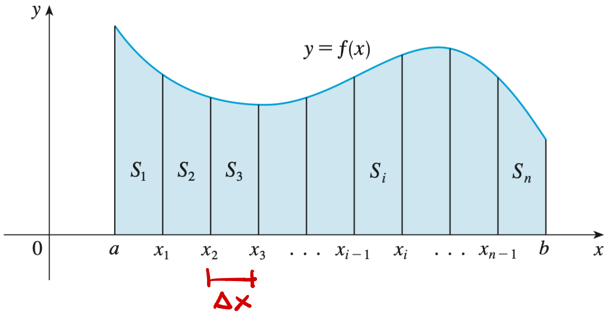
$$= \frac{(n+1)(2n+1)}{6n^2}$$

Hence $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{1}{6} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n}$

$$= \frac{1}{6} \cdot 1 \cdot 2 = \boxed{\frac{1}{3}}$$

✱ Note: we can show $\lim_{n\to\infty} L_n = \frac{1}{3}$ also

Example. Apply the idea of the previous example to estimate the area of the more general region below.



- Subdivide S into n strips of equal width $\frac{b-a}{n} \leftarrow \Delta x$

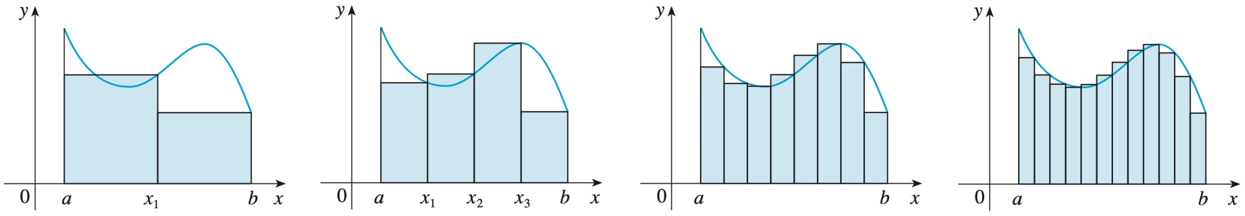
- The heights (using the value of f at the right endpoints) are:

$$f(x_1),\ f(x_2),\ \dots,\ f(x_{n-1}),\ f(\overset{x_n}{b})$$

- The total area is

$$\boxed{R_n = f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \dots + \overset{\text{height}}{f(x_n)}\cdot\overset{\text{base}}{\Delta x}}$$

- As $n \to \infty$, R_n approaches the actual area under $f(x)$.



(a) $n = 2$ (b) $n = 4$ (c) $n = 8$ (d) $n = 12$

Definition. Let f be continuous on an interval $[a,b]$. Divide $[a,b]$ into n equal parts of width $\Delta x = \dfrac{b-a}{n}$ with points $x_0 = a,\ x_1, \dots, x_n = b$. We have three ways to define the *area* A of the region under $y = f(x)$ from a to b.

1. Using right endpoints:

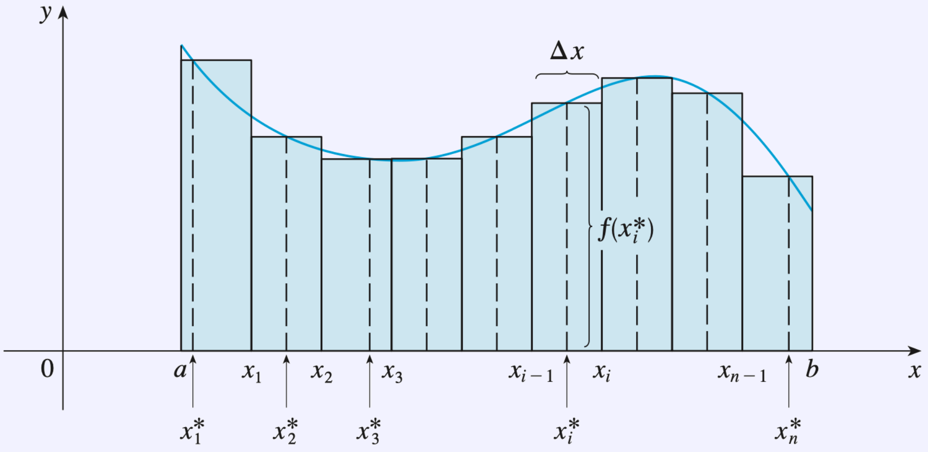
$$R_n = \big[\overset{\text{height}}{f(x_1)}\overset{\text{base}}{\Delta x} + f(x_2)\Delta x + \cdots + f(\overset{b}{x_n})\Delta x\big], \qquad A = \lim_{n\to\infty} R_n.$$

2. Using left endpoints:

$$L_n = \big[f(\overset{a}{x_0})\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x\big], \qquad A = \lim_{n\to\infty} L_n.$$

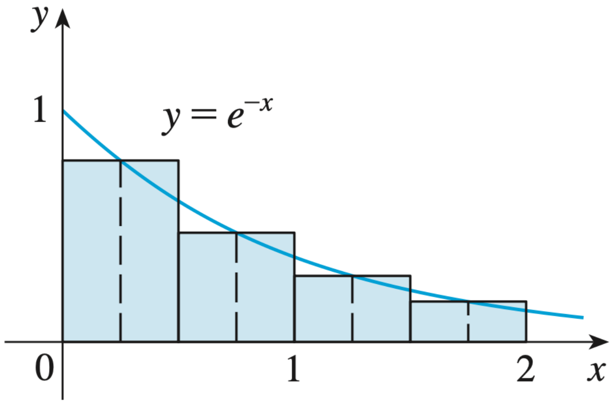
3. General sample points: in each subinterval $[x_{i-1}, x_i]$, choose any point x_i^*. Then,

$$S_n = \big[f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x\big], \qquad A = \lim_{n\to\infty} S_n.$$

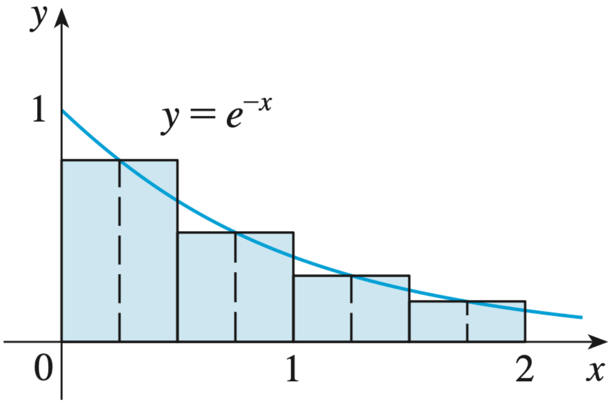


☆ **Theorem.** For a continuous function f, all three limits above exist and are equal; this common limit is the area A.

Example. Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$. Using **right endpoints**, find an expression for A as a limit.



Example. Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$. Estimate the area of A by taking the sample points to be **midpoints** and using four sub-intervals.



Example. Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds and record them in the following table:

Time (s)	0	5	10	15	20	25	30
Velocity (mi/h)	17	21	24	29	32	31	28
Velocity (ft/s)							