

→ Goal: Given f' , what is f ?

4.9 Antiderivatives

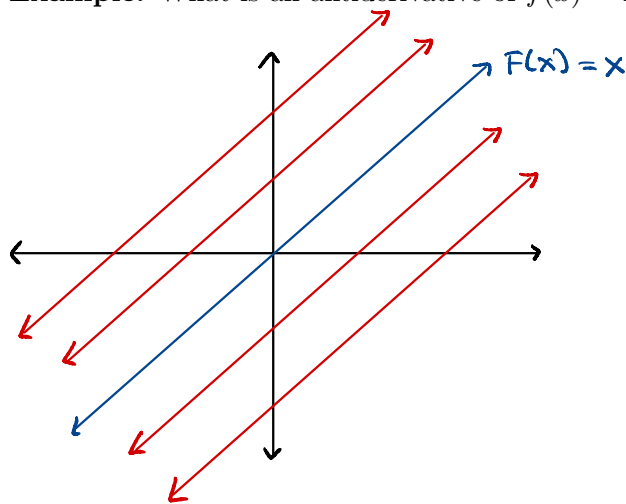
Definition. A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I .

Theorem. If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is

$$F(x) + C,$$

where C is an arbitrary constant.

Example. What is an antiderivative of $f(x) = 1$?



• If $F(x) = x$, then $F'(x) = 1$, so this is a particular antiderivative.

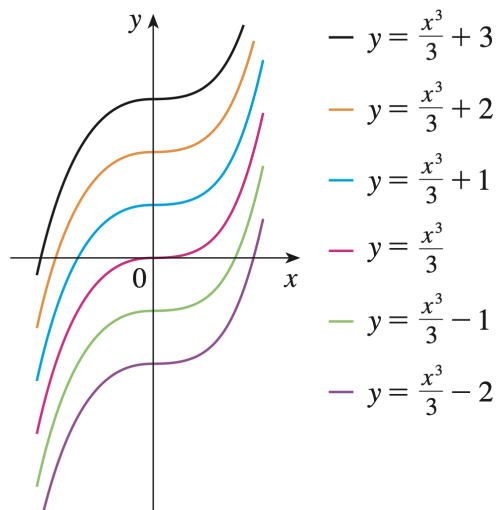
• But any line with slope 1 also has derivative $f(x) = 1$, e.g. $F(x) = x + 5$.

• We get a whole family of antiderivatives.

$$F(x) = x + C \quad \text{where } C \text{ is some constant.}$$

↪ The "most general" antiderivative.

Example. What is an antiderivative of $f(x) = x^2$?



$$- y = \frac{x^3}{3} + 3$$

$$- y = \frac{x^3}{3} + 2$$

$$- y = \frac{x^3}{3} + 1$$

$$- y = \frac{x^3}{3}$$

$$- y = \frac{x^3}{3} - 1$$

$$- y = \frac{x^3}{3} - 2$$

• $F(x) = \frac{1}{3}x^3$ is a particular antiderivative:

$$F'(x) = 3 \cdot \frac{1}{3}x^2 = x^2 \quad \checkmark$$

• Since we can add any constant to this, we get the whole family

$$F(x) = \frac{1}{3}x^3 + C$$

↪ The "most general" antiderivative.

Example. Find the most general antiderivative of $f(x) = x^n$ for any $n \neq -1$.

$F(x) = \frac{1}{n+1} \cdot x^{n+1}$ is a particular antiderivative.

$$\left[\text{Check: } F'(x) = (n+1) \cdot \frac{1}{n+1} x^n = x^n \checkmark \right]$$

The most general antiderivative is $F(x) = \frac{1}{n+1} \cdot x^{n+1} + C$

Example. Find the most general antiderivative of $f(x) = \sin(x)$.

$F(x) = -\cos(x)$ is a particular antiderivative:

$$\left[\text{Check: } F'(x) = -(-\sin(x)) = \sin(x) \checkmark \right]$$

The most general antiderivative is $F(x) = -\cos(x) + C$

Example. Find all functions g such that $g'(x) = 4 \sin x + \frac{2x^5 - \sqrt{x}}{x}$.

Rewrite this as $g'(x) = 4 \sin(x) + 2x^4 - x^{-1/2}$

By the above, $g(x) = 4(-\cos(x)) + 2 \cdot \frac{1}{5} x^5 - \frac{1}{1/2} x^{1/2} + C$

$$g(x) = -4 \cos(x) + \frac{2}{5} x^5 - 2\sqrt{x} + C$$

Example. Find f if $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$.

Rewrite this as $f'(x) = e^x + \frac{20}{1+x^2}$.

The most general antiderivative is $f(x) = e^x + 20 \tan^{-1}(x) + C$

Solve for C so that $f(0) = -2$:

$$-2 = e^0 + 20 \tan^{-1}(0) + C$$

$$-2 = 1 + 0 + C$$

$$-3 = C$$

$$f(x) = e^x + 20 \tan^{-1}(x) - 3$$

Example. Find f if $f''(x) = 12x^2 + 6x - 4$ if $f(0) = 4$ and $f(1) = 1$.

The most general antiderivative of $f''(x)$ is

$$f'(x) = 4x^3 + 3x^2 - 4x + C$$

The most general antiderivative of $f'(x)$ is

$$f(x) = x^4 + x^3 - 2x^2 + Cx + D$$

$$\text{Since } f(0) = 4 \Rightarrow D = 4$$

$$\text{Since } f(1) = 1$$

$$1 = 1^4 + 1^3 - 2 \cdot 1^2 + C \cdot 1 + 4$$

$$1 = 1 + 1 - 2 + C + 4$$

$$1 = C + 4$$

$$C = -3$$

$$f(x) = x^4 + x^3 - 2x^2 - 3x + 4$$

Example. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function $s(t)$.

The most general antiderivative of $a(t)$ is

$$v(t) = 3t^2 + 4t + C$$

Since $v(0) = -6 \Rightarrow C = -6$

$$v(t) = 3t^2 + 4t - 6$$

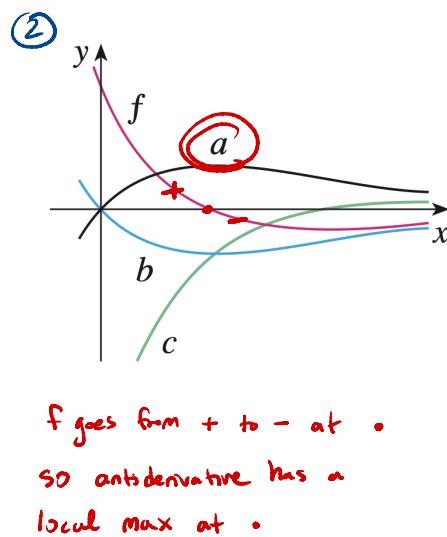
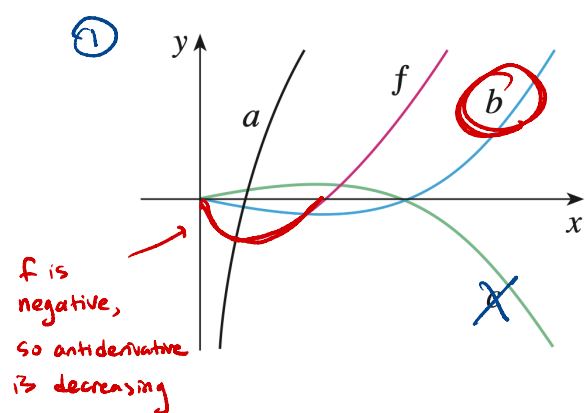
The most general antiderivative of $v(t)$ is

$$s(t) = t^3 + 2t^2 - 6t + D$$

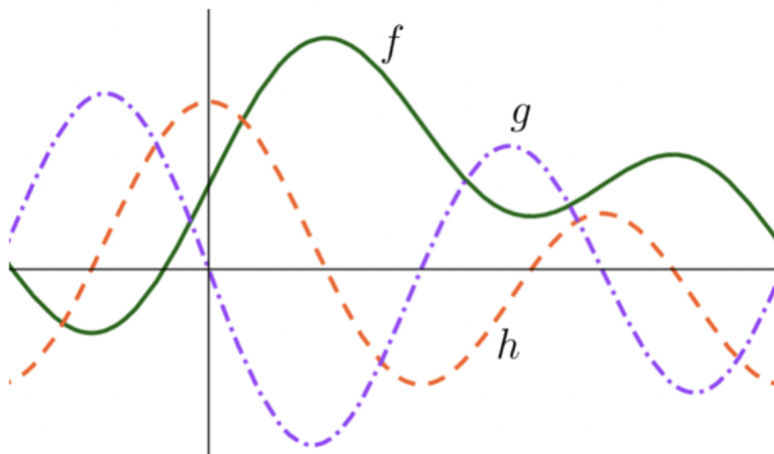
Since $s(0) = 9 \Rightarrow D = 9$

$$s(t) = t^3 + 2t^2 - 6t + 9$$

Example. The graph of a function f is shown. Which graph is an antiderivative of f and why?



Example. The graph of a function, its derivative, and one of its antiderivatives is pictured below.



- (a) f is an antiderivative of g and h is the derivative of g .
- (b) h is an antiderivative of g and f is the derivative of g
- (c) g is an antiderivative of f and h is the derivative of f
- (d) h is an antiderivative of f and g is the derivative of f
- (e) f is an antiderivative of h and g is the derivative of h