4.4 Indeterminate Forms and l'Hôpital's Rule

Indeterminate Differences

$\infty - \infty, \quad -\infty - (-\infty)$

Convert to a quotient using these options:

- combine using a common denominator
- rationalize (conjugate) when roots are present
- factor out the dominant term and compare

Indeterminate Products

$0 \cdot \infty,\ 0 \cdot (-\infty),\ \infty \cdot 0,\ (-\infty) \cdot 0$

Convert to a quotient:

$f \cdot g = \dfrac{f}{1/g} \quad \text{or} \quad f \cdot g = \dfrac{g}{1/f}.$

Indeterminate Powers

$0^0,\ \infty^0,\ 1^\infty$

Convert via logs: Let $y = f^{\,g}$. Then $\ln y = g \ln f$ (a product). Rewrite as a *quotient*; compute the limit, then exponentiate to get y.

Quotient forms eligible for l'Hôpital:

$\dfrac{0}{0}, \quad \dfrac{\infty}{\infty}, \quad \dfrac{-\infty}{\infty}, \quad \dfrac{\infty}{-\infty}, \quad \dfrac{-\infty}{-\infty}$

Apply l'Hôpital's Rule:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Guillaume François Antoine, Marquis de l'Hôpital
June 7, 1661 - February 2, 1704

Theorem (L'Hôpital's Rule)**.** Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0,$$

or

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty.$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

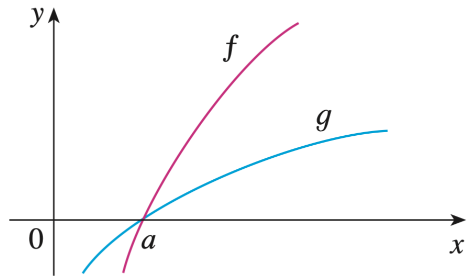
if the limit on the right side exists (or is ∞ or $-\infty$).

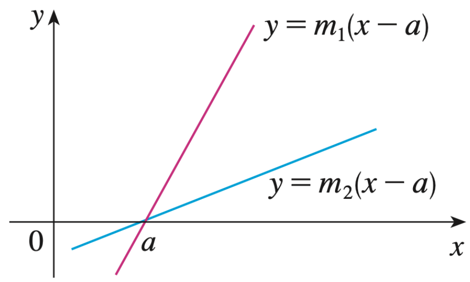
Proof. Prove the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$.

□

Remark. What about one-sided limits or limits at infinity?

Example. Use the graphs below to suggest why l'Hôpital's Rule might be true.





Example. Find $\lim_{x \to 1} \frac{\ln x}{x - 1}$.

Indeterminate Products ($0 \cdot \infty$ or $\infty \cdot 0$)

An indeterminate product occurs when, in a limit, one factor tends to 0 while the other tends to $\pm\infty$ (or vice versa). To evaluate such limits, convert the product into a quotient so that l'Hôpital's Rule can apply.

Procedure.

1. Rewrite fg as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$, producing a version of the form $0/0$ or ∞/∞.
2. Simplify algebraically (combine terms, rationalize, etc.).
3. Apply l'Hôpital's Rule as needed until the limit resolves.

Example. Evaluate $\lim_{x \to 0^+} x \ln x$.

Indeterminate Differences ($\infty - \infty$)

An indeterminate difference occurs when, in a limit, two terms both tend to $\pm\infty$ and we consider $f(x) - g(x)$. To evaluate such limits we first convert the difference into a form suitable for l'Hôpital's Rule (a $0/0$ or ∞/∞ quotient).

Procedure.

1. Combine terms to a single fraction (common denominator), rationalize with a conjugate, or factor out a common term to rewrite $f - g$ as a quotient that is $0/0$ or ∞/∞.
2. Simplify algebraically.
3. Apply l'Hôpital's Rule as needed until the limit resolves.

Example. Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$.

Indeterminate Powers ($0^0,\ \infty^0,\ 1^\infty$)

An indeterminate power arises in a limit of the form $\lim\limits_{x\to a}[f(x)]^{g(x)}$ when the base-exponent pair tends to one of:

$$\text{type } 0^0: \ f \to 0,\ g \to 0; \qquad \text{type } \infty^0: \ f \to \pm\infty,\ g \to 0; \qquad \text{type } 1^\infty: \ f \to 1,\ g \to \pm\infty.$$

To evaluate such limits, convert the power to a product (or quotient) so that l'Hôpital's Rule can be used.

Procedure.

1. If $f(x) > 0$ near a, set $y = [f(x)]^{g(x)}$, so that $\ln y = g(x)\ \ln f(x)$
2. Study $\lim\limits_{x\to a} g(x) \ln f(x)$ by converting to a quotient to obtain $0/0$ or ∞/∞.
3. Apply l'Hôpital's Rule as needed until the limit is found.
4. Conclude $\lim\limits_{x\to a}[f(x)]^{g(x)} = \lim\limits_{x\to a} e^{\ln y} = e^{\lim_{x\to a} \ln y}$.

Example. Calculate $\lim\limits_{x\to 0^+}(1 + \sin 4x)^{\cot x}$.