L'Hôpital's Rule — Differences, Products, and Powers

For each limit, classify the form: write the indeterminate type or write "determinate". Then evaluate the limit.

1. $\displaystyle\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x}$

This is type $\infty - \infty$. Combine into a single fraction:

$$\frac{1}{x} - \frac{1}{\sin x} = \frac{\sin x - x}{x\sin x},$$

which is $0/0$. Two applications of L'Hôpital:

$$\lim_{x\to 0^+} \frac{\sin x - x}{x\sin x} = \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} = \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0.$$

2. $\displaystyle\lim_{x\to\infty} xe^{-x}$

This is type $\infty \cdot 0$. Rewrite as a quotient and apply L'Hôpital:

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

3. $\displaystyle\lim_{x\to 0^+} \sqrt{x}\,\ln x$

This is type $0 \cdot (-\infty)$. Rewrite as a quotient:

$$\sqrt{x}\,\ln x = \frac{\ln x}{x^{-1/2}},$$

which is $-\infty/\infty$. L'Hôpital gives

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x\to 0^+} \frac{1/x}{-(1/2)x^{-3/2}} = \lim_{x\to 0^+} \left(-2x^{1/2}\right) = 0.$$

4. $\lim_{x\to 0^+} x^x$

This is type 0^0. Let $y = x^x$ and take logs:

$$\ln y = x \ln x = \frac{\ln x}{1/x} \quad \text{is } -\infty/\infty.$$

L'Hôpital:

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

Hence $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln(y)} = e^{\lim_{x\to 0^+} \ln(y)} = e^0 = 1.$

5. $\lim_{x\to 0^+} \left(1 + \sin x\right)^{1/x}$

This is type 1^∞. Let $y = (1 + \sin x)^{1/x}$ and take logs:

$$\ln y = \frac{\ln(1 + \sin x)}{x}.$$

Consider $L = \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(1 + \sin x)}{x}$, which is $0/0$. By L'Hôpital,

$$L = \lim_{x\to 0^+} \frac{\frac{\cos x}{1+\sin x}}{1} = 1.$$

Now exponentiate to return to y:

$$\lim_{x\to 0^+} \left(1 + \sin x\right)^{1/x} = \lim_{x\to 0^+} e^y = e^1 = e.$$