L'Hôpital's Rule — Type $0/0$ and ∞/∞ (Solutions)

A. For each limit, identify the *indeterminate form* by direct substitution. Write $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{-\infty}{\infty}$, $\frac{\infty}{-\infty}$, $\frac{-\infty}{-\infty}$, or determinate.

1. $\lim_{x\to 0} \dfrac{\sin x}{x}$ $\boxed{\dfrac{0}{0}}$

2. $\lim_{x\to 0} \dfrac{1-\cos x}{x}$ $\boxed{\dfrac{0}{0}}$

3. $\lim_{x\to \infty} \dfrac{\ln x}{\sqrt{x}}$ $\boxed{\dfrac{\infty}{\infty}}$

4. $\lim_{x\to \infty} \dfrac{e^{-x}}{x^3}$ $\boxed{\text{determinate}}$

5. $\lim_{x\to 0^+} \dfrac{\ln x}{1/x}$ $\boxed{\dfrac{-\infty}{\infty}}$

6. $\lim_{x\to 0^+} \dfrac{\csc x}{-\cot x}$ $\boxed{\dfrac{\infty}{-\infty}}$

B. Evaluate the following limits.

1. $\lim_{x\to 0} \dfrac{\sin(3x)}{x}$

   This is type $0/0$. By L'Hôpital,

   $$\lim_{x\to 0} \frac{\sin(3x)}{x} = \lim_{x\to 0} \frac{3\cos(3x)}{1} = 3.$$

2. $\lim_{x\to 1} \dfrac{x-\sqrt{x}}{x-1}$

   This is type $0/0$. By L'Hôpital,

   $$\lim_{x\to 1} \frac{x-\sqrt{x}}{x-1} = \lim_{x\to 1} \frac{1-\frac{1}{2\sqrt{x}}}{1} = 1 - \frac{1}{2} = \frac{1}{2}.$$

3. $\lim_{x\to \pi^-} \dfrac{\sin x}{1-\cos x}$

   This is determinate. Since $\sin(\pi) = 0$ and $1-\cos(\pi) = 2$,

   $$\lim_{x\to \pi^-} \frac{\sin x}{1-\cos x} = \frac{0}{2} = 0.$$

4. $\lim_{x\to 0} \frac{\tan x - x}{x^3}$

This is type 0/0. Apply L'Hôpital three times:

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \\ &= \lim_{x\to 0} \frac{2\sec^2 x\ \tan x}{6x} \\ &= \lim_{x\to 0} \frac{4\sec^2 x\ \tan^2 x + 2\sec^4 x}{6} = \frac{4\cdot 1\cdot 0 + 2\cdot 1}{6} = \frac{1}{3}.\end{aligned}$$

5. $\lim_{x\to 0} \frac{\ln(1+2x) - 2x}{x^2}$

This is type 0/0. Two applications of L'Hôpital:

$$\lim_{x\to 0} \frac{\ln(1+2x) - 2x}{x^2} = \lim_{x\to 0} \frac{\frac{2}{1+2x} - 2}{2x} = \lim_{x\to 0} \frac{-\frac{4}{(1+2x)^2}}{2} = -2.$$

6. $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$

This is type 0/0. Two applications of L'Hôpital:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \tfrac{1}{2}.$$

7. $\lim_{x\to\infty} \frac{e^x}{x^2}$

This is type ∞/∞. Two applications of L'Hôpital:

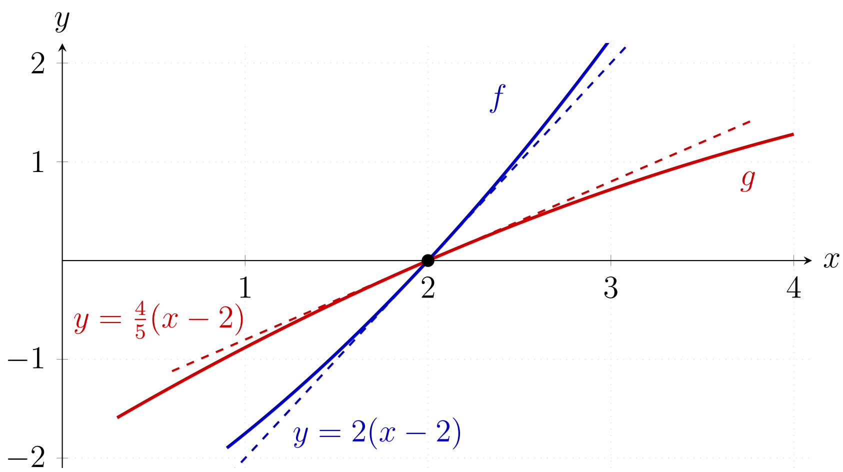
$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = +\infty.$$

8. $\lim_{x\to 0^+} \frac{\ln x}{\tan(x - \frac{\pi}{2})}$

This is type $\frac{-\infty}{-\infty}$. Two applications of L'Hôpital:

$$\lim_{x\to 0^+} \frac{\ln x}{\tan\left(x - \frac{\pi}{2}\right)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\sec^2\left(x - \frac{\pi}{2}\right)} = \lim_{x\to 0^+} \frac{\cos^2\left(x - \frac{\pi}{2}\right)}{x} = \lim_{x\to 0^+} \frac{2\cos(x - \frac{\pi}{2})}{1} = 0.$$

C. In the graph below, the blue curve is $f(x)$ and the red curve is $g(x)$. Both pass through $(2,0)$ and are differentiable at $x = 2$. The dashed lines are the tangent lines at $x = 2$. Evaluate $\lim_{x\to 2} \frac{f(x)}{g(x)}$.



$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \lim_{x\to 2} \frac{f'(x)}{g'(x)} = \frac{2}{4/5} = \frac{5}{2}.$$