New method for evaluating limits of "indeterminate forms"

4.4 Indeterminate Forms and l'Hôpital's Rule

Indeterminate Differences

$\infty - \infty, \quad -\infty - (-\infty)$

Convert to a quotient using these options:

- combine using a common denominator
- rationalize (conjugate) when roots are present
- factor out the dominant term and compare

Indeterminate Products

$0 \cdot \infty,\ 0 \cdot (-\infty),\ \infty \cdot 0,\ (-\infty) \cdot 0$

Convert to a quotient:

$f \cdot g = \dfrac{f}{1/g} \quad \text{or} \quad f \cdot g = \dfrac{g}{1/f}.$

Indeterminate Powers

$0^0,\ \infty^0,\ 1^\infty$

Convert via logs: Let $y = f^g$. Then $\ln y = g \ln f$ (a product). Rewrite as a *quotient*; compute the limit, then exponentiate to get y.

↓

Quotient forms eligible for l'Hôpital:

$\dfrac{0}{0},\ \dfrac{\infty}{\infty},\ \dfrac{-\infty}{\infty},\ \dfrac{\infty}{-\infty},\ \dfrac{-\infty}{-\infty}$

↓

Apply l'Hôpital's Rule:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Guillaume François Antoine, Marquis de l'Hôpital
June 7, 1661 - February 2, 1704

Theorem (L'Hôpital's Rule)**.** Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0,$$

or

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty.$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Proof. Prove the special case in which $f(a) = g(a) = 0$, f' and g' are continuous, and $g'(a) \neq 0$.

Continuity of f′ and g′ ; lim. def. of derivative

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \frac{\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\displaystyle\lim_{x\to a} \frac{g(x)-g(a)}{x-a}}$$

By quotient property of limits ; x−a terms cancel

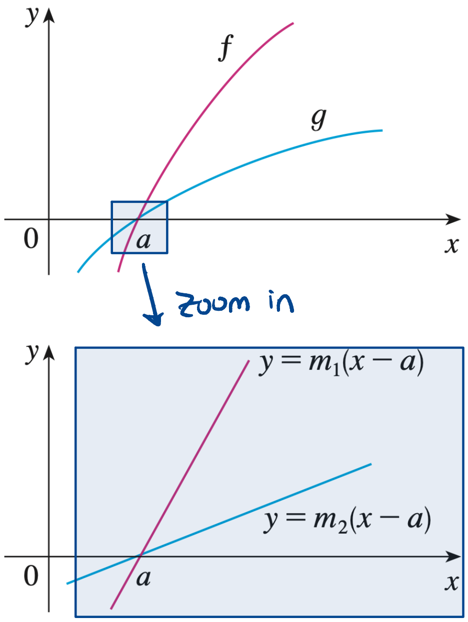
$$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{f(x)-\cancel{f(a)}^{\,0}}{g(x)-\cancel{g(a)}^{\,0}} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

□

Remark. What about one-sided limits or limits at infinity?

L'Hôpital's rule also holds for one-sided limits and limits at infinity. We can replace "$x\to a$" with "$x\to a^-$", "$x\to a^+$" or "$x\to \pm\infty$"

Example. Use the graphs below to suggest why l'Hôpital's Rule might be true.



- Note: both f and g approach 0 as $x \to a$

- To evaluate $\lim_{x \to a} \frac{f(x)}{g(x)}$, think about zooming in very close to a.

- If you zoom in far enough, both functions look linear

$\Rightarrow f(x) \approx m_1(x-a)$ (derivative of f at a)

$\Rightarrow g(x) \approx m_2(x-a)$ (derivative of g at a)

} the tangent lines

- $\lim_{x \to a} \frac{f(x)}{g(x)}$ "=" $\lim_{x \to a} \frac{m_1(x-a)}{m_2(x-a)} = \lim_{x \to a} \frac{m_1}{m_2} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Example. Find $\lim_{x \to 1} \dfrac{\ln x}{x - 1}$.

Since $\lim_{x \to 1} \ln(x) = 0$ and $\lim_{x \to 1} x - 1 = 0$, this is indeterminate of type $\frac{0}{0}$,

and so L'Hôpital's rule applies.

$$\lim_{x \to 1} \frac{\ln(x)}{x-1} = \lim_{x \to 1} \frac{1/x}{1} = \frac{1/1}{1} = \boxed{1}$$

Indeterminate Products ($0 \cdot \infty$ or $\infty \cdot 0$)

An indeterminate product occurs when, in a limit, one factor tends to 0 while the other tends to $\pm\infty$ (or vice versa). To evaluate such limits, convert the product into a quotient so that l'Hôpital's Rule can apply.

Procedure.

1. Rewrite fg as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$, producing a version of the form $0/0$ or ∞/∞.
2. Simplify algebraically (combine terms, rationalize, etc.).
3. Apply l'Hôpital's Rule as needed until the limit resolves.

Example. Evaluate $\lim_{x\to 0^+} x \ln x$.

This is type $0 \cdot (-\infty)$ since $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} \ln(x) = -\infty$

Writing x as $\dfrac{1}{1/x}$, we get

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{1/x}$$

← both terms go to $\pm\infty$ as $x \to 0^+$

By L'Hôpital,

$$\lim_{x\to 0^+} \frac{\ln(x)}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} \frac{1}{x}\cdot(-x^2) = \lim_{x\to 0^+} -x = 0$$

Indeterminate Differences ($\infty - \infty$)

An indeterminate difference occurs when, in a limit, two terms both tend to $\pm\infty$ and we consider $f(x) - g(x)$. To evaluate such limits we first convert the difference into a form suitable for l'Hôpital's Rule (a $0/0$ or ∞/∞ quotient).

Procedure.

1. Combine terms to a single fraction (common denominator), rationalize with a conjugate, or factor out a common term to rewrite $f - g$ as a quotient that is $0/0$ or ∞/∞.
2. Simplify algebraically.
3. Apply l'Hôpital's Rule as needed until the limit resolves.

Example. Compute $\lim_{x\to(\pi/2)^-} (\sec x - \tan x)$.

This is type $\infty - \infty$ since $\lim_{x\to\frac{\pi}{2}^-} \sec x = \infty$ and $\lim_{x\to\frac{\pi}{2}^-} \tan(x) = \infty$

Rewrite as:

$$\lim_{x\to\frac{\pi}{2}^-} \left[\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)} \right] = \lim_{x\to\frac{\pi}{2}^-} \frac{1-\sin(x)}{\cos(x)}$$

← This is type $\frac{0}{0}$

[L'Hôpital]

$$= \lim_{x\to\frac{\pi}{2}^-} \frac{-\cos(x)}{-\sin(x)}$$

$$= \frac{-\cos(\pi/2)}{-\sin(\pi/2)}$$

$$= \frac{0}{1}$$

$$= 0$$

Indeterminate Powers (0^0, ∞^0, 1^∞)

An indeterminate power arises in a limit of the form $\lim_{x\to a} [f(x)]^{g(x)}$ when the base-exponent pair tends to one of:

type 0^0 : $f \to 0,\ g \to 0$; type ∞^0 : $f \to \pm\infty,\ g \to 0$; type 1^∞ : $f \to 1,\ g \to \pm\infty$.

To evaluate such limits, convert the power to a product (or quotient) so that l'Hôpital's Rule can be used.

Procedure.

1. If $f(x) > 0$ near a, set $y = [f(x)]^{g(x)}$, so that $\ln y = g(x) \ln f(x)$
2. Study $\lim_{x\to a} g(x) \ln f(x)$ by converting to a quotient to obtain $0/0$ or ∞/∞.
3. Apply l'Hôpital's Rule as needed until the limit is found.
4. Conclude $\lim_{x\to a} [f(x)]^{g(x)} = \lim_{x\to a} e^{\ln y} = e^{\lim_{x\to a} \ln y}$.

Example. Calculate $\lim_{x\to 0^+} (1 + \sin 4x)^{\cot x}$.

This is type 1^∞ because $\lim_{x\to 0^+} (1+\sin 4x) = 1$ and $\lim_{x\to 0^+} \cot x = \infty$

Let $y = (1+\sin 4x)^{\cot x}$ $\Rightarrow$ $\ln(y) = \ln\left[(1+\sin 4x)^{\cot x}\right]$

$\Rightarrow$ $\ln(y) = \cot x \cdot \ln(1+\sin 4x)$ ← This is type $\infty \cdot 0$

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan(x)} \overset{\text{L'Hôpital}}{=} \lim_{x\to 0^+} \frac{\frac{4\cos(4x)}{1+\sin(4x)}}{\sec^2(x)} = \frac{\frac{4}{1}}{1} = \boxed{4}$$

Hence,

$$\lim_{x\to 0^+} (1+\sin(4x))^{\cot(x)} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln(y)} = e^{\lim_{x\to 0^+} \ln(y)} = \boxed{e^4}$$